Uncertainty: A Reading Guide and Self-Paced Tutorial

First, read the description of uncertainty at the link *Uncertainty Review*, up to and including Rule 6, making sure you understand what is meant by "measurement uncertainty", and "worst case analysis". Now you are ready to test your understanding by working through this tutorial. As you read along, think about the questions, and fill in the numbered blank spaces. You can check your answers against the "key" on the last page.

We begin by reviewing the concept of **measurement uncertainty**. To start practicing the cases described in the link *Uncertainty Review*, pretend you want to measure the thickness of your physics text. As a first try, you use a plastic ruler from the bookstore. The markings on the ruler are 0.1 cm apart, the markings themselves appear to be 0.01 or 0.02 cm thick, and the first (0 cm) mark seems to be 0.01 cm away from the end of the ruler. You lay the book face up on a desk, stand the ruler on end alongside the book, and look to see how the top of the book lines up with the ruler markings. It lines up beautifully with the 5.0 cm mark. Now, test your judgment by filling in the blanks before looking at the answer key. With this information, you are justified in saying that the measurement uncertainty is:

measurement uncertainty = __1__.

The thing to remember about this is that there isn't a unique answer! Your uncertainty estimate will to some extent reflect how conservative a person you are. If you knew beforehand that someone with a very precise measuring device had measured the book's thickness, and you'd get a $500 reward if your uncertainty range included the precision measurement, you'd probably make a larger uncertainty estimate. In any event, your estimate should have been one of those in the answer key. If you aren't sure why, ask when you come to lab.

In the second approach of Appendix A-1, you measure the thickness of the book as before, but at different places around the edge of the book, perhaps at each corner and at the center of each side. The measurements give the following results, where estimates for hundredths are included:

5.02 cm, 5.10 cm, 5.10 cm, 5.26 cm, 5.31 cm, 5.21 cm, 5.29 cm, 5.20 cm

With this information, you should record the "measurement uncertainty" as:

measurement uncertainty = __2__.

This time the answer in the key is slightly on the conservative side. You would be justified in giving an answer half as large. Remember, uncertainties aren't exact: they just indicate the "order of magnitude" to which a result is believable. For this reason, you should always state uncertainties to one "significant figure", e.g. you could answer the last question with either 0.2 cm or 0.1 cm, but not 0.15 cm.

The results of the two approaches above are typical. The thickness obtained from a single measurement in the first approach turned out to be at one end of the range of values measured in the second, and the measurement uncertainty estimated in the first approach was significantly smaller than that in the second. For this reason, we ask you to "justify" your measurement uncertainties by using the second method, i.e. make at least three independent measurements of the quantity, taking turns with your lab partner, and trying not to remember the last result when finding the next. This sounds like extra work, but it rarely takes more than a minute, and remember, we only require this "three trials, alternating partners" (or "3XAP") approach the first time you make a particular kind of measurement. Thereafter, you may simply use the same measurement uncertainty, justifying it by referring back to this first instance by giving its page number.

The final scenario in Appendix A-1 can be illustrated by imagining that you're measuring the period of a pendulum, the time it takes to go through one complete swing, from left to right and back to left again. You use an electronic timing circuit that displays to the nearest 0.01 sec. You and your lab partner take turns making the following five measurements:

2.00 sec, 2.00 sec, 2.00 sec, 2.00 sec, 2.00 sec

Since all measurements gave the same result, the measurement uncertainty is ______[3]. This is because the smallest scale division on the measuring instrument is 0.01 sec.

When you know the uncertainty in a measurement, you can write the result with the correct number of **significant figures**. This refers to the digits in a numerical result, i.e. the digit 3 in the result 932.47 is the tens digit, while the 7 is the hundredths digit. Suppose you measure something once, and obtain 932.47 meters as the result. You then repeat the measurement many times, and find that the hundreds digit (9) and the tens digit (3) always come out the same. However, the units digit (2) is a 1 in 25% of the measurements, a 2 in 50%, and a 3 in the remaining 25%, and the last two digits (tenths and hundredths) vary randomly over all the possibilities from 0 to 9. In this case the first two digits are definitely significant, because they have the same value in every measurement. The last two digits are insignificant, because the measurements don't tell you anything about their actual values. The units digit is borderline; it's not completely random but not completely reproducible either. We call it significant however, because the series of measurements does narrow it down quite a bit. It's definitely somewhere between a 1 and a 3, and is most probably a 2. Significant figures are the digits that are at least partly predictable after a series of trial measurements.

In this manual, we use the words "**final result**" to mean the result of a series of at least five trials, written so that only significant figures appear, with the measurement uncertainty and appropriate units included. The final result of the measurements in the previous paragraph would be 932 meters ± 1 meter, or (932 ± 1) meters.

Now we move on to **worst case analysis**. Suppose you want to determine the speed of a jogger by measuring the time spent jogging a certain distance. The results of your measurements are:

distance = (100.0 ± .5) meters
time = (36.2 ± .1) sec

Using speed = distance/time, a worst case analysis of uncertainty for the jogger's speed looks like:

most probable ("best") speed = (______[4])/(36.2 sec) = 2.762430939 m/s

largest possible ("max") speed = (100.5m)/(___5___) = 2.783933518 m/s
smallest possible ("min") speed = (___6___)/(___7___) = 2.741046832 m/s

Did you notice that to get the "max" speed you had to put the "min" time in the denominator, and vice-versa? Some people are startled by this, but if you think about what you're doing it makes perfect sense! Speed increases if it takes a shorter time to cover the distance. The above analysis yields the following "final result" for the speed:

speed = (2.76 ± .02) m/s.

Note that in the calculation of the max, min, and best speeds, it's OK to write a ridiculously large number of figures after the decimal point, because it is only by comparing these three values that we decide how many of these figures are significant. Since the uncertainty in the speed is about two hundredths of a meter per second, it would clearly not be sensible to specify the thousandths digit or any of the digits farther to the right of the decimal. That's why the final result is written with only the two significant digits after the decimal point. Remember, the uncertainty part of the final result has only one significant figure, and the number of decimal places is the same in both the final value and its uncertainty. Another thing to notice is the set of parentheses written in the final result. These indicate that the units, m/s, apply to both the speed, 2.76 m/s, and its uncertainty, 0.02 m/s. You could write speed = 2.76 m/s ± .02 m/s, but it would be incorrect to write speed = 2.76 m/s ± .02 or to write speed = 2.76 ± .02 m/s.

Suppose that as a good experimentalist, you wanted to reduce the uncertainty in your result. Obviously, you need to reduce the measurement uncertainties in the distance and time. But which of these is most important? To answer this question, we want to find out how much of the final uncertainty in the speed is due to the distance measurement, and how much is due to the time measurement. We therefore consider each measurement separately, a technique called partial uncertainty analysis. To begin, let's ignore the uncertainty in time, and focus on the uncertainty in distance. We calculate the speed twice, using the "best" value for the time in both calculations, but using the max distance in one and the min distance in the other:

1) "best" time, "max" distance:

speed = 100.5 m / 36.2 s = 2.776243094 m/s

2) "best" time, "min" distance:

speed = 99.5 m / 36.2 s = 2.748618785 m/s

The difference between these two results is 0.027624309 m/s. Because best values are usually about halfway between the corresponding max and min values, we divide this difference in half to find out the amount of uncertainty in the speed that is caused just by the distance measurement:

Partial uncertainty in speed due to the distance measurement = .013812154 m/s
= .014 m/s.

To find the partial uncertainty due just to the time measurement, we ignore the uncertainty in distance, holding the distance constant at its best value, but varying the time from max to min:

3) max time, best distance:

speed = 100 m / 36.3 s = 2.754820937 m/s

4) min time, best distance:

speed = 100 m / 36.1 s = 2.770083102 m/s

As we saw earlier, putting in the max time gives a min speed, and vice-versa, since time is in the denominator. But we don't care about that, we just want to know the range over which the speed varies when the time is varied from max to min. Half this range is the partial uncertainty we seek:

Partial uncertainty in speed due to the time measurement = .007631082 m/s
= .008 m/s

Now compare the two partial uncertainties. The one due to distance, .014 m/s, is almost twice as large as the .008 m/s due to time. That tells us that it makes more sense to try to reduce the uncertainty in the distance measurement than to work on improving the time measurement. Of course, the sum of the two partial uncertainties, .008 m/s + .014 m/s = .022 m/s, the same total uncertainty we got in our original worst case analysis.

Why calculate partial uncertainties instead of doing it all at once as in our original worst case analysis? One reason is so we can see which measurement contributes the most uncertainty to the final result. Another is that in many cases you can't decide whether to put the max or the min into the formula when you're trying to calculate the "worst case". As an example, look at the following formula, which you will use in Lab O-3:

$$R = \frac{(L^2 + d^2)}{2d}$$

Here R is calculated from measurements of L and d. Suppose you were doing a worst case analysis, and wanted to calculate the largest possible R. Obviously, you would plug in the max L because L is in the numerator. But what about d? Since d appears in both the numerator and the denominator, it's very difficult to see in advance whether you should use its max or min value get the biggest R. By doing partial uncertainties, you avoid having to deal with this kind of puzzle.

Now you can practice what you've learned, using the above formula for R. Suppose you have just measured the following:

L = (25.4 ± 0.2) mm.
d= (2.0 ± 0.1) mm

First, we find the best R, by plugging the best values of L and d into the formula:
Best R = (______) [8]

Next, let's do the trickier of the two partial uncertainties, the one with respect to d. We plug in the best L, vary d from max to min, and note the change in R:

1) max d, best L:

R = [(25.4)2 + (____9____)2] / [(2) (____10____)] mm
= 154.6595238 mm

Some people get confused with this calculation, because they wonder if they should insert the max d in the numerator and the min d in the denominator, or if they should put the max d in both places. But if you remember that we're asking "What would R be if d were 2.1 mm instead of 2.0 mm?", the answer should be clear. When d is 2.1 mm, it is 2.1 mm in the numerator and also in the denominator - it's impossible for d to be 2.1 mm and 1.9 mm at the same time! Let's continue our search for the partial uncertainty in R with respect to d:

2) min d, best L:

R = [(25.4)2 + (____11____)2] / [(2) (____12____)] mm
= 170.7289474 mm.

To get the partial uncertainty, we take the ____13____, divide by ____14____, and get rid of at least the wildly insignificant figures, with the result that the partial uncertainty in R with respect to d = 8 mm.

Next, we pretend there's no uncertainty in d, and look at the effect of the uncertainty in L:

3) best d, max L: R = (____15____)

4) best d, min L: R = (____16____)

This tells us that the measurement of (____17____) causes the most uncertainty in our final result for R, and that the final result for R is:

R = [(____18____) ± (____19____)] mm.

Are you surprised by the answer in the key? Perhaps you expected it to say something like R = 162 mm ± 8 mm. That isn't a terribly unreasonable answer, but if the uncertainty were 18 mm you would surely round it up to 20mm and say R = (160 ± 20) mm, so you should really round the 8 mm up to 10 mm as well. The cases you'll find hardest to decide are the ones like 134 ± 8, or 135 ± 7! Do you say 130 ± 10, 140 ±10, or what? Probably the best answer is to leave them as they are, without rounding the uncertainty up to 10, but don't become paranoid about it - we'll be sympathetic in situations like this! But we'll also go berserk if we see something like 134.3279 ± 6.6342771!

If you're afraid that you're in for lots of tedious and repetitive calculations, take heart! It turns out that the procedure described above is really simple and fast if you do it in Excel, the software package provided on the lab computers and on the campuswide network. The tutorial you are reading now is just meant to explain what partial uncertainties are, and they do involve repetitive calculations, but that's exactly the kind of thing at which computers are great, so take the time to learn about Excel, and you'll find that partial uncertainty analysis is a snap!

The last thing to learn about uncertainty is the standard deviation approach. This method is based on probability theory, which we don't want to get into here. Take a look now at the last part of Appendix A-1 in the lab manual, without paying too much attention to the formula you see there. Note that when you take a really (infinitely) large number of measurements, the results will be distributed around the average value as shown by a "bell-shaped curve", and that 68.3% of your results will fall within one standard deviation, σ, of the average. That means that if you make a single measurement, there is a 68.3% chance that it will fall within $\pm\,\sigma$ of the average, and you can say that the uncertainty in one measurement is the standard deviation.

Although individual measurements "jump around" by $\pm\,\sigma$, the average of **N** measurements is much more stable. According to "Rule 11" of Appendix A-1, the uncertainty in the average is given by $\sigma/\sqrt{N}$, if N is large. Since this is an introductory lab, we'll be content to say that "large **N**" means **N** $\geq$ 10! So in this lab, if you measure something at least 10 times, you can say that (1) the most probable result is simply the average of the data values, and (2) the uncertainty in the average is the standard deviation divided by the square root of the number of measurements. Naturally, the Excel tutorial will teach you how to find standard deviations in a very simple and speedy way.

In conclusion, you can say that your "final result" for a measurement that you've made at least 10 times is:

"final result" = (average value) ± (standard deviation/$\sqrt{N}$),

where **N** is the number of trials.

Answer Key:

1. 0.01 cm, 0.02 cm, or 0.03 cm.
2. 0.2 cm.
3 0.01 s
4. 100.0 m
5. 36.1 s
6. 99.5 m
7. 36.3 s
8. 162.29 mm
9. 2.1
10. 2.1
11. 1.9
12. 1.9
13. difference
14. 2
15. 164.84 mm
16. 159.76 mm
17. d
18. 160
19. 10