Lesson Seven: Math Miscellany

A consequence of the Pythagorean Theorem is the fact that $\sin^2\theta + \cos^2\theta = 1$.

One of the most amazing integrals is the following. Who'd have thought that π would pop in there?!?

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$$

And can you believe what happens when you sum the reciprocals of all the perfect squares? What on earth could that have to do with circles? And yet π makes another appearance...

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{\pi^2}{6}$$

Some of our favorite numbers are $e = \lim_{n\to\infty} (1 + 1/n)^n$ and $\ln 2 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$.

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Definition. The *inverse tangent* or *arctangent* is denoted $\tan^{-1} x$ or $\arctan x$ and is the angle between $-\pi/2$ and $\pi/2$ whose tangent is equal to x.

Example 1. Since $\tan(\pi/4) = 1$, we also have $\arctan 1 = \pi/4$.

Example 2. However, $\tan(9\pi/4) = 1$, but $\arctan 1 = \pi/4$.

Question to Ponder 1. Should we really be anthropomorphizing an angle by using the word "whose" here?

Now we use the arctangent and some calculus to derive a wonderful series.

$\frac{\pi}{4} = \arctan 1$

$= \int_0^1 \frac{1}{1+x^2}\,dx$	Surely you remember that $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$.
$= \int_0^1 \frac{1}{1-(-x^2)}\,dx$	Prepare to use the series $\frac{1}{1-u} = 1 + u + u^2 + u^3 + \ldots$
$= \int_0^1 [1 - x^2 + x^4 - \ldots]\,dx$	Substitute $u = -x^2$ into the series above. Duh.
$= \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots\right]_0^1$	Ever heard of the FTC?
$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$	A moment of silence, please.

Question to Ponder 2. What on earth do the reciprocals of the odd natural numbers have to do with π?!?