Lesson Nine: Arrows and Functions

We have a wide range of arrows in our quiver, a small sample of which appears below.

$$\Rightarrow \quad \leftarrow \quad \longleftarrow \quad \uparrow \quad \Downarrow \quad \swarrow \quad \mapsto \quad \leftrightarrow$$

Definition. A function $f\colon A \to B$ is *injective* or *one-to-one* if and only if whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$.

Notation. If f is injective, we write $f\colon A \xrightarrow{\text{1-1}} B$.

Definition. A function $f\colon A \to B$ is *surjective* or *onto* if and only if for every $b \in B$, there exists $a \in A$ such that $f(a) = b$.

Notation. If f is surjective, we write $f\colon A \xrightarrow[\text{onto}]{} B$.

Some books use the symbol $\stackrel{\text{def}}{=}$ when defining a function. For instance, we have the **identity function** on a set A, which sends each element of A to itself.

$$I_A(x) \stackrel{\text{def}}{=} x$$

Exercise Nine: Arrows and Functions

Definition. A function $f\colon A \to B$ is *bijective* or a *one-to-one correspondence* if and only if f is injective and surjective.

Notation. If f is a bijection, we write $f\colon A \xrightarrow[\text{onto}]{\text{1-1}} B$.

Claim 1. *The function $f\colon \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = x^4 + 1$ is not injective.*

Proof. We must show that there exist $a_1, a_2 \in \mathbb{R}$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$.

Choose $a_1 = 1$ and $a_2 = -1$.

Then $f(a_1) = 2$ and $f(a_2) = 2$ but $a_1 \neq a_2$, so f is not injective.

□

Claim 2. *The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 1$ is surjective.*

Proof. We must show that for every $b \in \mathbb{R}$ [the codomain] there exists $a \in \mathbb{R}$ [the domain] such that $f(a) = b$.

Pick any $b \in \mathbb{R}$.

Let $a = (b-1)/3$.

Since $b \in \mathbb{R}$ and because the reals are closed under subtraction and non-zero division, we know that $(b-1)/3 \in \mathbb{R}$, *i.e.*, a is in the domain of f.

Furthermore,

$$\begin{aligned} f(a) &= f\left(\frac{b-1}{3}\right) \\ &= 3 \cdot \frac{b-1}{3} + 1 \\ &= b - 1 + 1 \\ &= b \end{aligned}$$

as desired. □