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Solution of the Basic Problems of Electrodynamics in the Group-Space Formulation (*).

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Summary. — The question is pursued of whether there exists a *complete* alternate formulation of classical electrodynamics in terms of a *single scalar* function, which may then lead to a new formulation of quantum electrodynamics. Using harmonic analysis and spin-weighted spherical harmonics we study such a theory and solve it for three deliberately simple but characteristic problems: point charges, current loops and antennae. The connection between the field strength and potential formulations is established, as is the relation of this theory to the conventional one using vector spherical harmonics.

PACS. 03.50.De – Maxwell theory: general mathematical aspects.

1. – Introduction.

As the history of physics shows, new formulations of existing theories extend the scope of these theories and provide deeper insight into the phenomena

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they describe. Well-known examples of this are the Newtonian, Lagrangian and Hamiltonian formulations of classical mechanics, and the Schrödinger, Heisenberg and Feynman formulations of quantum mechanics. Another important example is provided by electrodynamics. In this case, the theory can be formulated in terms of the field strengths $\boldsymbol{E}$ and $\boldsymbol{B}$, or in terms of the potentials A_μ. The potential formulation is prefered in quantum theory and quantum field theory because the coupling of the electromagnetic field to the matter field Ψ is easily formulated in terms of the A_μ and because of the occurrence of gauge-covariant derivatives $(\partial_\mu - eA_\mu)$. The A_μ themselves, however, contain unwanted degrees of freedom and thus overdetermine the electromagnetic field since different potentials may describe the same physical situation. The field strengths, on the other hand, seem to underdetermine the electromagnetic field since, as the Aharonov-Bohm effect (1,2) shows, phase factors of the form $\exp[(iq/\hbar c)\oint A_\mu \, \mathrm{d}x^\mu]$ are observable in multiple connected regions where $\boldsymbol{E}$ and $\boldsymbol{B}$ vanish (3). The description of such phenomena in terms of field strengths is at least very complicated. Thus it continues to be important to search for new formulations of electrodynamics that will represent, perhaps more faithfully, the physical phenomena, or at least simplify the description.

It has been known for a long time that Maxwell's equations can be rewritten in terms of $\boldsymbol{\Psi} = \boldsymbol{E} + i\boldsymbol{B}$ as a Dirac-like equation. This equation seems to be rediscovered many times (4). But $\boldsymbol{\Psi}$ here is a spinorlike quantity. We are interested in a scalar representation of the electromagnetic field. Such a representation has been obtained when the longitudinal part of $\boldsymbol{A}$ is eliminated (5). Another scalar representation, which we study here, is obtained by applying the method of writing vector and tensor fields as functions over the group SU_2 (6). Then, by the methods of harmonic analysis, the fields can be expanded in the basis provided by the matrix elements of the irreducible representations of SU_2. The underlying mathematics is well known and has been used in general relativity (7), as well as in multipole expansions (8). When

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this method is applied to the radial and radial-helicity components of $(\boldsymbol{E} + i\boldsymbol{B})$, a simplification occurs and the problem of solving Maxwell's equations reduces to that of solving *one* partial differential equation for *one* complex scalar function [9,10]. Since the number of independent variables (real) is equal to the number of degrees of freedom, the quantization of the electromagnetic field free from gauge problems becomes possible [10]. Thus this alternative formulation which, at least formally, seems to better represent the electromagnetic field, should be further studied.

The same method has been used in other areas. In 1974 it was used to reformulate the linearized equations of general relativity and was shown to result in a gauge-free quantization of the linearized gravitational field [11]. More recently, it has led a new Hilbert space for quantum gravity that is also applicable to a wide range of Riemannian space-times [12-14]. It has been used to reformulate electromagnetic-scattering problems [15], the field equations of the Weyl and Dirac fields [16], and has also been applied to the equations for the A_μ. In this latter case one again obtains a single differential equation for one complex scalar function, and the number of independent real variables is equal to the number of degrees of freedom [17,18]. Although the gauge choice is not completely eliminated, the Lorentz gauge emerges as the prefered one, just as it does in the covariant Green's function method of field theory, and a quantization of the electromagnetic field results that is free of all gauge problems [18].

The purpose of the present paper is to unify and complete the group space formulation of classical electrodynamics by connecting the field strength and potential approaches, both with sources, and to present the general solutions for both. The general solutions are derived in sect. **2** and **3**. In sect. **4** the connection of the two formulations is given. In sect. **5** we illustrate the general solution for three basic sources of electrodynamics: a point charge, a stationary current loop and a dipole antenna. These are simple prototypes of many other sources. As a step towards the reformulation of the Maxwell-Dirac equations of quantum electrodynamics, we indicate, at the end of sect. **2**, how the harmonic analysis can be extended to the remaining (r, t) variables.

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2. – Solving for the electromagnetic potential in the SU_2 basis.

In two previous papers (17,18) we expanded the electromagnetic potentials in a basis of functions provided by the matrix elements of the irreducible representations of SU_2, and derived the differential equations determining the expansion coefficients. In this section we solve these equations and show explicitly how the expansion coefficients are determined from the sources, both in static and time-varying cases.

In keeping with past notation (17,18) we represent the scalar part of the electromagnetic potential by A_0 and the radial part by A_r. If we define

$$A_\pm = -(A_\varphi \pm iA_\theta)/\sqrt{2}\,, \tag{2.1}$$

then, as we show in appendix A, $(\pm iA_\pm)$ are the components of the electromagnetic potential along the positive and negative (radial) helicity vectors $(\hat{e}_\theta \pm i\hat{e}_\varphi)/\sqrt{2}$. Moreover, as was discussed in ref. (9), we can define an angle φ_2 describing the orientation of the helicity vectors in the plane perpendicular to $\hat{r}$.

Given these definitions, we introduce four functions:

$$\xi_t = A_0\,, \tag{2.2}$$

$$\xi_0 = A_r \tag{2.3}$$

and

$$\xi_\pm = A_\pm \exp[\mp i\varphi_2]\,. \tag{2.4}$$

As was discussed in ref. (6), these functions have particularly simple expansions in the basis provided by the matrix elements T^j_{nm} of the irreducible representations of SU_2:

$$\xi_t = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \hat{a}^j_{0m}(t, r)\, T^j_{0m}(u)\,, \tag{2.5}$$

$$\xi_0 = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} a^j_{0m}(t, r)\, T^j_{0m}(u) \tag{2.6}$$

and

$$\xi_\pm = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} a^j_{\pm1,m}(t, r)\, T^j_{\pm1,m}(u)\,. \tag{2.7}$$

The expansion coefficients in eqs. (2.5)-(2.7) are found from the orthogonality relations

$$\int \mathrm{d}u\, T^j_{mn}(u)\, T^{j'}_{m'n'}(u)^* = (2j+1)^{-1}\delta_{jj'}\delta_{mm'}\delta_{nn'}\,, \tag{2.8}$$

where $\mathrm{d}u$ is the invariant measure over SU_2:

$$\mathrm{d}u = (16\pi)^{-1} \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi_1\, \mathrm{d}\varphi_2\,, \tag{2.9}$$

$\varphi_1 = \pi/2 - \varphi$, and the limits of integration are $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi_1 \leqslant 4\pi$, $0 < \varphi_2 < 2\pi$. Thus

$$\hat{a}^j_{0m} = (2j+1)\int \xi_t(t, r, u)\, T^j_{0m}(u)^* \, du\,, \tag{2.10}$$

$$a^j_{0m} = (2j+1)\int \xi_0(t, r, u)\, T^j_{0m}(u)^* \, du \tag{2.11}$$

and

$$a^j_{\pm 1,m} = (2j+1)\int \xi_\pm(t, r, u)\, T^j_{\pm 1,m}(u) \, du\,. \tag{2.12}$$

Similarly we can expand the charge density ϱ and the current density $\boldsymbol{J}$ as

$$\varrho(\boldsymbol{r}, t) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} \varrho^j_{0m}(t, r)\, T^j_{0m}(u)\,, \tag{2.13}$$

$$J_r(\boldsymbol{r}, t) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} J^j_{0m}(t, r)\, T^j_{0m}(u) \tag{2.14}$$

and

$$\tilde{J}_\pm(\boldsymbol{r}, t) = -\,(J_\varphi \pm iJ_\theta)\exp[\mp i\varphi_2]/\sqrt{2}\,, \tag{2.15}$$

$$\tilde{J}_\pm(\boldsymbol{r}, t) = J_\pm \exp[\mp i\varphi_2]\,, \tag{2.16}$$

$$\tilde{J}_\pm(\boldsymbol{r}, t) = \sum_{j=1}^{\infty} \sum_{m=-j}^{j} J^j_{\pm 1,m}(t, r)\, T^j_{\pm 1,m}(u)\,. \tag{2.17}$$

The expansion coefficients are found from the equations

$$\varrho^j_{0m} = (2j+1)\int \varrho(t, r, u)\, T^j_{0m}(u)^* \, du\,, \tag{2.18}$$

$$J^j_{0m} = (2j+1)\int J_r(t, r, u)\, T^j_{0m}(u)^* \, du \tag{2.19}$$

and

$$J^j_{\pm 1,m} = (2j+1)\int \tilde{J}_\pm(t, r, u)\, T^j_{\pm 1,m}(u)^* \, du\,. \tag{2.20}$$

We showed previously that, when the electromagnetic potential is restricted to the Lorentz gauge, the expansion coefficients are determined from differential equations [18] in the remaining (t, r)-variables:

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{j(j+1)}{r^2}\right](r\hat{a}^j_{0m}) = r\varrho^j_{0m}\,, \tag{2.21}$$

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{j(j+1)}{r^2}\right](r^2 a^j_{0m}) = r^2 J^j_{0m} + 2\frac{\partial}{\partial t}(r\hat{a}^j_{0m})\,, \tag{2.22}$$

$$\left[\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2} + \frac{j(j+1)}{r^2}\right](r a^j_{\pm 1,m}) = rJ^j_{\pm 1,m} + \sqrt{2j(j+1)}\, a^j_{0m}/r\,, \tag{2.23}$$

when Maxwell's equations are expressed in Heaviside-Lorentz units with $c = 1$.

The solutions to these equations must further satisfy the Lorentz condition [17,18]

$$\frac{\partial}{\partial t}(r\hat{a}^j_{0m}) + \frac{1}{r}\frac{\partial}{\partial r}(r^2 a^j_{0m}) + \sqrt{\frac{j(j+1)}{2}}\,[a^j_{+1,m} - a^j_{-1,m}] = 0\,. \tag{2.24}$$

If we wish to find the electromagnetic potential when no sources are present, then eqs. (2.21)-(2.23) show that we need only to solve one partial differential equation, eq. (2.21), for one complex-valued function $(r\hat{a}^j_{0m})$. The other three expansion coefficients are then determined from this one by eqs. (2.22)-(2.24). Thus the problem of solving for the electromagnetic potential in the Lorentz gauge reduces to that of solving for one complex scalar function ξ_t [17,18]. This demonstrates the formal advantage of the SU_2 approach: namely that the number of independent real variables is equal to the number of degrees of freedom. We will see in the next section that the same situation holds when the field $(\boldsymbol{E} + i\boldsymbol{B})$ is expanded in the T^j_{nm} basis [10]. Thus the SU_2 formulation provides a way to quantize the free electromagnetic field, in terms of the field strengths themselves or the potentials, that is free of spurious degrees of freedom.

If sources are present, then it is convenient to recast eqs. (2.21)-(2.23) as integral equations from which the expansion coefficients can be determined more readily. Because any time-dependent source can be expanded by using a Fourier analysis, it is sufficient to consider a single Fourier component with frequency ω:

$$\varrho(\boldsymbol{r}, t) = \varrho(\boldsymbol{r}) \exp[-i\omega t]\,, \tag{2.25}$$

$$\boldsymbol{J}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}) \exp[-i\omega t]\,, \tag{2.26}$$

where, as usual, the physical situation is obtained by taking real parts. Assuming the sources have the above time dependence implies that the potentials have the same time dependence:

$$\begin{aligned} \hat{a}^j_{0m}(t, r) &= \hat{a}^j_{0m}(r) \exp[-i\omega t]\,, \\ a^j_{0m}(t, r) &= a^j_{0m}(r) \exp[-i\omega t]\,, \\ a^j_{\pm 1,m}(t, r) &= a^j_{\pm 1,m}(r) \exp[-i\omega t] \end{aligned}$$

and that the spatial parts are determined by the equations

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \omega^2 - \frac{j(j+1)}{r^2}\right](r\hat{a}^j_{0m}) = -r\varrho^j_{0m}\,, \tag{2.27}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \omega^2 - \frac{j(j+1)}{r^2}\right](r^2 a^j_{0m}) = -r^2 J^j_{0m} + 2i\omega r\hat{a}^j_{0m}\,, \tag{2.28}$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \omega^2 - \frac{j(j+1)}{r^2}\right](ra^j_{\pm 1,m}) = -rJ^j_{\pm 1,m} - \sqrt{2j(j+1)}\,a^j_{0m}/r\,, \tag{2.29}$$

together with the subsidiary Lorentz condition

$$(2.30)\qquad -i\omega r\hat{a}^j_{0m} + \frac{1}{r}\frac{\partial}{\partial r}(r^2 a^j_{0m}) + \sqrt{\frac{j(j+1)}{2}}\,(a^j_{1,m} - a^j_{-1,m}) = 0\,.$$

The differential equations (2.27)-(2.29), plus the boundary conditions of finiteness at the origin and outgoing waves at infinity, are equivalent to the integral equations

$$(2.31)\qquad \hat{a}^j_{0m}(r) = -r^{-1}\int\limits_0^\infty \mathrm{d}r'\, g_j(r, r')\, r'\, \varrho^j_{0m}(r')\,,$$

$$(2.32)\qquad a^j_{0m}(r) = -r^{-2}\int\limits_0^\infty \mathrm{d}r'\, g_j(r, r')[r'^2 J^j_{0m}(r') - 2i\omega r' a^j_{0m}(r')]\,,$$

$$(2.33)\qquad a^j_{\pm 1,m}(r) = -r^{-1}\int\limits_0^\infty \mathrm{d}r'\, g_j(r, r')[r' J^j_{\pm 1,m}(r') + \sqrt{2j(j+1)}\, a^j_{0m}(r')/r']\,.$$

For sources oscillating with frequency ω the Green's functions satisfying the boundary conditions can be written as [19]

$$(2.34)\qquad g_j(r, r') = -\omega^{-1}\hat{\jmath}_j(\omega r_<)\,\hat{h}^+_j(\omega r_>)\,,$$

while for time-independent sources ($\omega = 0$) eq. (2.34) reduces to

$$(2.35)\qquad g_j(r, r') = -(2j+1)^{-1} r_<^{j+1}/r_>^j$$

with $r_<$ and $r_>$ being, respectively, the smaller and larger of r and r'. The functions $\hat{\jmath}$ and $\hat{h}^+$ are the Riccati-Bessel and Riccati-Hankel functions

$$(2.36)\qquad \hat{\jmath}_l(z) \equiv z j_l(z) = (\pi z/2)^{\frac{1}{2}} J_{l+\frac{1}{2}}(z)\,,$$

$$(2.37)\qquad \to \frac{z^{l+1}}{(2l+1)!!} \qquad (\text{as } z \to 0)\,,$$

$$(2.38)\qquad \to \sin(z - l\pi/2) \qquad (\text{as } z \to \infty)$$

and

$$(2.39)\qquad \hat{h}^+_l(z) = \hat{n}_l(z) + i\hat{\jmath}_l(z)\,,$$

$$(2.40)\qquad \to (2l-1)!!\, z^{-l} \qquad (\text{as } z \to 0)\,,$$

$$(2.41)\qquad \to \exp[i(z - l\pi/2)] \qquad (\text{as } z \to \infty)\,.$$

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[19] J. R. TAYLOR: *Scattering Theory* (Wiley, New York, N. Y., 1972).

Equations (2.31)-(2.33), together with the Green's functions (2.34) and (2.35), give the explicit solutions for the expansion coefficients for any source distribution.

Before ending this section, we note that in many cases of interest the observation point lies outside the sources and thus eqs. (2.31)-(2.33) can be written as

$$\hat{a}^j_{0m}(r) = \frac{\hat{h}^+_j(\omega r)}{\omega r}\int_0^\infty \mathrm{d}r'\,\hat{\jmath}_j(\omega r')\,r'\,\varrho^j_{0m}(r')\,, \tag{2.42}$$

$$a^j_{0m}(r) = \frac{\hat{h}^+_j(\omega r)}{\omega r^2}\int_0^\infty \mathrm{d}r'\,\hat{\jmath}_j(\omega r')[r'^2 J^j_{0m}(r') - 2i\omega r'\,\hat{a}^j_{0m}(r')]\,, \tag{2.43}$$

$$a^j_{\pm1,m}(r) = \frac{\hat{h}^+_j(\omega r)}{\omega r}\int_0^\infty \mathrm{d}r'\,\hat{\jmath}_j(\omega r')[r' J^j_{\pm1,m}(r') + \sqrt{2j(j+1)}\,a^j_{0m}(r')/r']\,, \tag{2.44}$$

when the sources are oscillating with frequency ω, and for time-independent sources, as

$$\hat{a}^j_{0m}(r) = (2j+1)^{-1} r^{-j-1}\int_0^\infty \mathrm{d}r'\,r'^{j+2}\,\varrho^j_{0m}(r')\,, \tag{2.45}$$

$$a^j_{0m}(r) = (2j+1)^{-1} r^{-j-2}\int_0^\infty \mathrm{d}r'\,r'^{j+3}\,J^j_{0m}(r')\,, \tag{2.46}$$

$$a^j_{\pm1,m}(r) = (2j+1)^{-1} r^{-j-1}\int_0^\infty \mathrm{d}r'\,r'^{j+1}[r' J^j_{\pm1,m}(r') + \sqrt{2j(j+1)}\,a^j_{0m}(r')/r']\,. \tag{2.47}$$

Finally, in the radiation zone ($\omega r \gg 1$, $\omega r \gg j(j+1)$), the function $\hat{h}^+$ can be replaced in eqs. (2.42)-(2.44) by its asymptotic form (2.41).

Remark. The operators

$$\mathscr{L}_\pm = r\left[-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{j(j+1)}{r^2} \pm \omega^2\right] \quad \text{and} \quad T = -i\left[r\frac{\partial}{\partial r} + 1\right]$$

generate a $SU_{1,1}$ dynamical group. Thus, in eqs. (2.27)-(2.29), we could perform another harmonic analysis in the remaining variables (r, t) using the irreducible representations of $SU_{1,1}$, in the same spirit as the harmonic analysis in the (θ, φ) variables. We then would use the unitary irreducible representations of $SU_{1,1}$ on the r-variable instead of the Green's function method used here to establish contact with the standard results.

3. – Solving for $(\boldsymbol{E}+i\boldsymbol{B})$ in the SU_2 basis.

In 1970 CARMELI [10] expanded the radial and radial-helicity components of $(\boldsymbol{E}+i\boldsymbol{B})$ in the SU_2 basis and derived the differential equations determining the expansion coefficients. In this section we review his work and recast the differential equations for the expansion coefficients into integral equations. We also establish the connection between the expansion coefficients of $(\boldsymbol{E}+i\boldsymbol{B})$ in the SU_2 basis and the more conventional multipole coefficients associated with the vector spherical harmonics.

In keeping with with Carmeli's notation we define $\boldsymbol{V}=(\boldsymbol{E}+i\boldsymbol{B})$ and

$$\eta_0 = V_r\,, \qquad \eta_\pm = -(V_\varphi \pm i V_\theta)\exp[\mp i\varphi_2]/\sqrt{2}\,. \tag{3.1}$$

Because of the way they are defined, the eta-functions can be expanded in the SU_2 basis as

$$\eta_0 = \sum_{j=0}^{\infty}\sum_{m=-j}^{j}\alpha^j_{0m}(t,r)\,T^j_{0m}(u)\,, \tag{3.2}$$

$$\eta_\pm = \sum_{j=1}^{\infty}\sum_{m=-j}^{j}\alpha^j_{\pm1,m}(t,r)\,T^j_{\pm1,m}(u) \tag{3.3}$$

with expansion coefficients given by

$$\alpha^j_{\pm1,m}(t,r) = (2j+1)\int\eta_0(t,r,u)\,T^j_{0m}(u)^*\,\mathrm{d}u\,, \tag{3.4}$$

$$\alpha^j_{\pm1,m}(l,r) = (2j+1)\int\eta_\pm(t,r,u)\,T^j_{\pm1,m}(u)^*\,\mathrm{d}u\,. \tag{3.5}$$

When expansions (3.2) and (3.3) are substituted into Maxwell's equations with sources (expressed in Heaviside-Lorentz units with $c=1$), the expansion coefficients are determined by the equations

$$\left[\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial r^2}+\frac{j(j+1)}{r^2}\right](r^2\alpha^j_{0m}) = \\ = -\left\{\left(\frac{\partial}{\partial r}\pm\frac{\partial}{\partial t}\right)r^2(\varrho^j_{0m}\pm J^j_{0m})+\sqrt{2j(j+1)}\,r\,J^j_{\pm1,m}\right\} \tag{3.6}$$

and

$$\alpha^j_{\pm1,m} = \sqrt{2j(j+1)}\left\{\frac{1}{r}\left(\frac{\partial}{\partial t}\mp\frac{\partial}{\partial r}\right)(r^2\alpha^j_{0m})\pm r(\varrho^j_{0m}\pm J^j_{0m})\right\}, \tag{3.7}$$

when $j = 1, 2, 3, \ldots; \ m = -j, -j+1, \ldots, j$; and

$$\frac{\partial}{\partial t}\alpha^0_{00} = -J^0_{00}\,, \tag{3.8}$$

$$\frac{\partial}{\partial r}(r^2\alpha^0_{00}) = r^2\varrho^0_{00}\,, \tag{3.9}$$

when $j = m = 0$.

If we wish to find $(\boldsymbol{E} + i\boldsymbol{B})$ when no sources are present, then eqs. (3.6) and (3.7) show that the functions α^j_{0m} completely determine the $\alpha^j_{\pm1,m}$, *i.e.* the radial component of $(\boldsymbol{E} + i\boldsymbol{B})$ completely determines the radial helicity components. Thus the problem of solving Maxwell's equations reduces to that of finding one scalar complex function η_0. As mentioned in sect. **2**, this shows that in the SU_2 approach the number of independent variables is equal to the actual number of degrees of freedom and provides a way to quantize the field strengths themselves that is entirely gauge free ([10]).

If sources are present, we can assume they have the time dependence (2.25) and (2.26), in which case the solution to eq. (3.6) becomes

$$\alpha^j_{0m}(r) = -r^{-2}\int\limits_0^\infty \mathrm{d}r'\, g_j(r, r')\left\{\left(\frac{\partial}{\partial r'} \mp i\omega\right) r'^2(\varrho^j_{0m} \pm J^j_{0m}) + \sqrt{2j(j+1)}\,r' J^j_{\pm1,m}\right\} \tag{3.10}$$

with

$$\alpha^j_{\pm1,m} = -[2j(j+1)]^{-\frac{1}{2}}\left\{\frac{1}{r}\left(\pm\frac{\partial}{\partial r} + i\omega\right)(r^2\alpha^j_{0m}) \mp r(\varrho^j_{0m} \pm J^j_{0m})\right\}. \tag{3.11}$$

The Green's function used in eq. (3.10) is again that of eq. (2.34) for time-dependent sources and (2.35) for time-independent sources.

For observation points outside the sources eqs. (3.10) and (3.11) can be written as

$$\alpha^j_{0m}(r) = \frac{\hat{h}^+_j(\omega r)}{\omega r^2}\int\limits_0^\infty \mathrm{d}r'\, \hat{\jmath}_j(\omega r')\left\{\left(\frac{\partial}{\partial r'} \mp i\omega\right) r'^2(\varrho^j_{0m} \pm J^j_{0m}) + \sqrt{2j(j+1)}\,r' J^j_{\pm1,m}\right\} \tag{3.12}$$

and

$$\alpha^j_{\pm1,m}(r) = -[2j(j+1)]^{-\frac{1}{2}}r^{-1}\left(\pm\frac{\partial}{\partial r} + i\omega\right)(r^2\alpha^j_{0m})\,. \tag{3.13}$$

If the observation point lies in the radiation zone, $\hat{h}^+$ can be replaced by its asymptotic form (2.41).

For time-independent sources and observation points lying outside the source region, eq. (3.10) and (3.11) can be written as

$$(3.14)\qquad \alpha^j_{0m}(r) = (2j+1)^{-1} r^{-j-2}\int_0^\infty \mathrm{d}r'\, r'^{j+1}\left\{\frac{\partial}{\partial r'} r'^2(\varrho^j_{0m} \pm J^j_{0m}) + \sqrt{2j(j+1)}\, r' J^j_{\pm 1,m}\right\}$$

and

$$(3.15)\qquad \alpha^j_{\pm 1,m}(r) = \mp\, [2j(j+1)]^{-\frac{1}{2}} r^{-1} \frac{\partial}{\partial r}\,(r^2\, \alpha^j_{0m})\,.$$

It is worth noting that eq. (3.12) can be written in a form more suitable for applications by making two modifications. First, since the Riccati-Bessel function $\hat{j}$ is zero at the origin and the sources are finite, we can integrate by parts to show that

$$(3.16)\qquad \int_0^\infty \mathrm{d}r'\, \hat{j}_j(\omega r')\, \frac{\partial}{\partial r'}\,(r'^2 \varrho^j_{0m}) = -\int_0^\infty \mathrm{d}r'(r'^2 \varrho^j_{0m})\, \frac{\partial}{\partial r'}\, \hat{j}_j(\omega r')\,.$$

Second, we can incorporate the restriction that conservation of current

$$(3.17)\qquad \nabla\cdot \boldsymbol{J} + \partial\varrho/\partial t = 0$$

places on the expansion coefficients. Proceding in the same way as the derivation of (30) of ref. [18], we find

$$(3.18)\qquad -\,i\omega r \varrho^j_{0m} + \frac{1}{r}\,\frac{\partial}{\partial r}\,(r^2 J^j_{0m}) + \sqrt{\frac{j(j+1)}{2}}\,[J^j_{+1,m} - J^j_{-1,m}] = 0\,.$$

Using eqs. (3.18) and (3.16) in eq. (3.12) we further obtain

$$(3.19)\qquad \alpha^j_{0m}(r) = -\,\frac{\hat{h}^+_j(\omega r)}{\omega r^2}\int_0^\infty \mathrm{d}r'\left\{(r'^2\varrho^j_{0m})\,\frac{\partial \hat{j}_j(\omega r')}{\partial r'} + i\omega r'^2\,\hat{j}_j(\omega r')\, J^j_{0m} - \right.$$
$$\left. -\, r'\,\hat{j}_j(\omega r')\sqrt{\frac{j(j+1)}{2}}\,[J^j_{+1,m} - J^j_{-1,m}]\right\}.$$

Thus the expansion coefficients for $(\boldsymbol{E} + i\boldsymbol{B})$ in the SU_2 basis are determined by eqs. (3.19) and (3.15) when the observation point is outside of the sources.

Equation (3.19) is quite similar to the standard expressions for the electric- and magnetic-multipole coefficients which occur when the fields are expanded in vector spherical harmonics [20]. Indeed, by using definitions (2.36) and

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[20] J. D. JACKSON: *Classical Electrodynamics* (Wiley, New York, N. Y., 1975).

(2.39) and the expression

$$\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J}) = -\boldsymbol{r}\cdot\nabla\times\boldsymbol{J} = \frac{-i}{\sqrt{2}}(K_-J_+ + K_+J_-)\,,$$

where [15]

$$K_\pm T^j_{mn} = [(j\pm m+1)(j\mp m)]^{\frac{1}{2}}\, T^j_{m\pm 1,n}$$

with

$$K_\pm = \exp[\mp i\varphi_2]\left(\pm \operatorname{ctg}\theta\,\frac{\partial}{\partial\varphi_2} + i\,\frac{\partial}{\partial\theta} \mp \operatorname{cosec}\theta\,\frac{\partial}{\partial\varphi_1}\right),$$

the relation to multipole coefficients is easily established. An alternative and simpler way is to note that the electric- and magnetic-multipole coefficients are defined on p. 747 of ref. [20] to be

$$a_M(j,m)\,h_j^{(1)}(\omega r) = \omega[j(j+1)]^{-1}\int Y^*_{jm}\, rB_r\,\mathrm{d}\Omega \tag{3.20}$$

and

$$a_E(j,n)\,h_j^{(1)}(\omega r) = -\,\omega[j(j+1)]^{-1}\int Y^*_{jm}\, rE_r\,\mathrm{d}\Omega \tag{3.21}$$

for observation points outside the sources. Equations (3.20) and (3.21) can be combined to show that

$$-\frac{\sqrt{j(j+1)}}{\omega r}\,h_j^{(1)}(\omega r)(a_E - i a_M) = \int Y^*_{jm}\eta_0\,\mathrm{d}\Omega\,. \tag{3.22}$$

Using expansion (3.2) and the result [15]

$$T^j_{0m} = \sqrt{\frac{4\pi}{2j+1}}\,Y_{jm} \tag{3.23}$$

in eq. (3.22) we get finally

$$\alpha^j_{0m}(r) = -\sqrt{\frac{2j+1}{4\pi}}\,\sqrt{j(j+1)}\,\frac{h_j^{(1)}(\omega r)}{\omega r}\,(a_E - i a_M)\,, \tag{3.24}$$

which is the desired relation between the two sets of expansion coefficients.

4. – Connecting the field strength and potential expansions in the SU_2 basis.

In sect. **2** and **3** we discussed how the field strengths $(\boldsymbol{E}+i\boldsymbol{B})$ and the potentials $(A_0, \boldsymbol{A})$ could be expanded in the SU_2 basis and found the integral equations determining the expansion coefficients. In this section we relate the two approaches, so that, once the expansion coefficients for the potentials are known, the expansion coefficients for the field strengths can be determined from them directly.

We begin by deriving an expression for α^j_{0m} in terms of the a^j_{nm}. The defining relations for the potentials are

$$B = \nabla \times A \tag{4.1}$$

and

$$E = -\nabla A_0 - \frac{\partial A}{\partial t}. \tag{4.2}$$

Since α^j_{0m} is the expansion coefficient for the radial component of $(E + iB)$, we first find the radial components of eqs. (4.10) and (4.2) and then express them in terms of the ξ-functions. From eq. (4.2) and (2.2), (2.3) we have

$$E_r = -\frac{\partial}{\partial r}\xi_t - \frac{\partial}{\partial t}\xi_0 \tag{4.3}$$

and from eq. (4.1) and (2.4)

$$B_r = \frac{i}{\sqrt{2}\,r}(K_-\xi_+ + K_+\xi_-)\,, \tag{4.4}$$

where the $K_\pm$ operators are defined after eq. (3.19). Equations (4.3) and (4.4) together imply that

$$\eta_0 = -\frac{\partial\xi_t}{\partial r} - \frac{\partial\xi_0}{\partial t} - \frac{1}{\sqrt{2}\,r}(K_-\xi_+ + K_+\xi_-)\,. \tag{4.5}$$

Substituting expansion (2.5)-(2.7) and (3.2) into eq. (4.5) and equating coefficients gives the desired result

$$\alpha^j_{0m} = -\left\{\frac{\partial \hat{a}^j_{0m}}{\partial r} + \frac{\partial a^j_{0m}}{\partial t} + \sqrt{\frac{j(j+1)}{2}}\,\frac{(a^j_{1,m} + a^j_{-1,m})}{r}\right\}. \tag{4.6}$$

The expansion coefficients $\alpha^j_{\pm 1,m}$ can be determined from eq. (4.6) and (3.9) or from eq. (4.1) and (4.2) The first method will relate the expansion coefficients for the radial helicity components of $(E + iB)$ to the expansion coefficients for the potentials *and* the sources, while the second will involve only the potentials. To pursue the second method, we note that

$$\eta_\pm = -\frac{1}{\sqrt{2}}\{(E_\varphi \pm iE_\theta) + i(B_\varphi \pm iB_\theta)\}\exp[\pm i\varphi_2]\,. \tag{4.7}$$

Taking angular components of (4.1) and (4.2), we find

$$-\frac{1}{\sqrt{2}}(E_\varphi \pm iE_\theta)\exp[\mp i\varphi_2] = \pm\frac{1}{\sqrt{2}\,r}K_\pm\xi_t - \frac{\partial\xi_t}{\partial t} \tag{4.8}$$

and

$$-\frac{1}{\sqrt{2}}(B_\varphi \pm iB_\theta)\exp[\mp i\varphi_2] = \pm\frac{1}{r}\frac{\partial}{\partial r}(r\xi_\pm) + \frac{1}{\sqrt{2}\,r}K_\pm\xi_0 . \tag{4.9}$$

Substituting (4.8) and (4.9) into (4.7) and using the appropriate expansions gives the desired result

$$\alpha^j_{\pm1,m} = \frac{1}{r}\left(\pm\frac{\partial}{\partial r} - \frac{\partial}{\partial t}\right)(ra^j_{\pm1,m}) + \sqrt{\frac{j(j+1)}{2}}\left(\frac{a^j_{0m} \pm \hat{a}^j_{0m}}{r}\right). \tag{4.10}$$

Equations (4.6) and (4.10) allow the expansion coefficients for the field strengths to be computed directly from the expansion coefficients for the potentials once the latter are known. There is still some gauge freedom left in eqs. (4.6) and (4.10), namely

$$\hat{a}^j_{0m} \to \hat{a}^j_{0m} + \frac{\partial\Lambda}{\partial t}, \qquad a^j_{0m} \to a^j_{0m} - \frac{\partial\Lambda}{\partial r},$$

$$ra^j_{\pm1,m} \to ra^j_{\pm m} + \left(\frac{\partial\Lambda}{\partial r} \pm \frac{\partial\Lambda}{\partial t}\right)$$

with

$$\Box\Lambda \equiv \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right)\Lambda = 0 .$$

5. – Solving dynamical problems with sources.

We illustrate now the formalism developed in sect **2-4** by solving for the fields created by three fundamental sources: the electrostatic point charge —the fundamental problem of electrostatics, the circular loop carrying a steady current—the fundamental problem of magnetostatics, and the center-fed, linear antenna—a fundamental problem in radiation theory.

A) The electrostatic point charge. Without loss of generality we assume that our electric charge q is located at the origin. In this case the source densities are

$$\varrho(\boldsymbol{r}) = q\delta(\boldsymbol{r})/4\pi r^2, \qquad \boldsymbol{J}(\boldsymbol{r}) = 0 .$$

Our first step in applying the SU_2 formalism is to expand these sources in the SU_2 basis and determine the expansion coefficients. Using eqs. (2.8), (2.18) and (3.23), we find that

$$\varrho^j_{0m} = q\delta(r)\,\delta_{j0}\delta_{m0}/4\pi r^2 \tag{5.1}$$

and $J^j_{0m} = J^j_{\pm 1,m} = 0$ for all j, m.

To find the electric and magnetic fields created by this source, we use eqs. (3.8) and (3.9) which imply that

$$\alpha^0_{00} = q/4\pi r^2 .$$

Thus, using eq. (3.23), we find

$$(\boldsymbol{E} + i\boldsymbol{B})_r = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} \alpha^j_{0m} T^j_{0m} = \alpha^0_{00} T^0_{00} = q/4\pi r^2 . \tag{5.2}$$

Hence

$$E_r = q/4\pi r^2 . \tag{5.3}$$

All other components of the field strengths are zero. This is the familiar expression for the electric field from a point charge q located at the origin, expressed in Heaviside-Lorentz units.

We can also find the potential created by this source and use our connection formulae to derive the field strengths. Using eqs. (2.45)-(2.47), we find

$$\hat{a}^j_{0m} = q\delta_{j0}\delta_{m0}/4\pi r \tag{5.4}$$

and $a^j_{0m} = a^j_{\pm 1,m} = 0$ for all j and m. Putting these into eqs. (2.5)-(2.7) implies that

$$A_0 = q/4\pi r \tag{5.5}$$

and $A_r = A_\pm = 0$. This is just the scalar potential from a point charge located at the origin, expressed in Heaviside-Lorentz units. Finally, using the connection formulae (4.6), we find

$$\alpha^0_{00} = -\frac{\partial}{\partial r}(q/4\pi r) = q/4\pi r^2 , \tag{5.6}$$

as we found directly in eq. (5.2).

B) *The circular current loop.* As our next example we consider a circular current loop of radius a, lying in the (x, y)-plane centered at the origin, carrying a steady current I. The charge density ϱ is zero as are all the components of the current density J except (see ref. [20], p. 177)

$$J_\varphi = I\delta(\cos\theta)\,\delta(r-a)/a\,. \tag{5.7}$$

Using this in eq. (2.15), we find

$$J_\pm = -\,I\delta(\cos\theta)\,\delta(r-a)\exp[\mp i\varphi_2]/\sqrt{2}\,a\,. \tag{5.8}$$

By expanding this source distribution in the SU_2 basis according to eq. (2.17) the expansion coefficients are found from eq. (2.20) to be

$$J^j_{\pm1,m} = -\,\frac{(2j+1)}{16\pi^2}\,\frac{I\delta(r-a)}{\sqrt{2}a}\cdot \int_{-1}^{1}\mathrm{d}(\cos\theta)\int_0^{4\pi}\mathrm{d}\varphi_1\int_0^{2\pi}\mathrm{d}\varphi_2\,\delta(\cos\theta)\exp[\mp\, i\varphi_2]\,T^j_{\pm1,m}(n)^*\,. \tag{5.9}$$

From eqs. (A.17) and (A.18) we know that $T^j_{\pm1,m}$ is proportional to $\exp[-\,im\varphi_1]$ and $\exp[\mp i\varphi_2]$, thus the integral over φ_1 gives a factor of $4\pi\delta_{m0}$ (reflecting the cylindrical symmetry of the source), the integral over φ_2 gives a factor of 2π, and the integral over θ forces $\theta = \pi/2$. Thus, using eqs. (A.24) and (A.25), we find

$$J^j_{\pm1,m} = \frac{iI(2j+1)}{2\sqrt{2j(j+1)}\,a}\,\delta_{m0}\,\delta(r-a)\,P^1_j(0)\,, \tag{5.10}$$

where $P^m_j(x)$ is the usual associated Legendre polynomial. $P^1_j(0)$ is evaluated on p. 179 of ref. [20]. It is zero if j is even (reflecting the parity of the source) and, for j odd, it is given by

$$P^1_j(0) = \frac{(-1)^{n+1}\Gamma(n+3/2)}{\Gamma(n+1)\Gamma(3/2)}\,, \tag{5.11}$$

where $j = 2n+1$ and $n = 0, 1, 2, \ldots$. Equation (5.11) can be simplified by using standard properties of the gamma-function to read

$$P^1_{2n+1}(0) = \frac{(-1)^{n+1}(2n+1)!!}{2^n n!}\,, \tag{5.12}$$

for $n = 0, 1, 2, \ldots$.

To find the electromagnetic potential when $r > a$, we use eqs. (5.10) in (2.74):

$$a^j_{\pm1,m} = \frac{iI\delta_{m0}}{2\sqrt{2j(j+1)}}\left(\frac{a}{r}\right)^{j+1}P^1_j(0)\,. \tag{5.13}$$

All the other expansion coefficients are zero. Putting (5.13) into (2.7), we find

$$\xi_\pm = -\frac{I}{2\sqrt{2}}\exp[\mp i\varphi_2]\sum_{n=0}^{\infty}\frac{P'_{2n+1}(0)}{(2n+1)(2n+2)}\left(\frac{a}{r}\right)^{2n+2}P'_{2n+1}(\cos\theta)\,. \tag{5.14}$$

In this equation we have used eq. (A.24) from the appendix to express $T^j_{\pm 1,0}$ in terms of the associated Legendre polynomials. Equations (5.12) and (2.4) give then the desired result

$$A_\varphi = -\frac{Ia}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n-1)!!}{(n+1)!2^n}\frac{1}{r}\left(\frac{a}{r}\right)^{2n+1}P'_{2n+1}(\cos\theta)\,, \tag{5.15}$$

which is eq. (5.46) of ref. [20] expressed in Heaviside-Lorentz units with $c=1$. (The unit conversion is accomplished by replacing c^{-1} in Gaussian units by $(4\pi c)^{-1}$ [20].)

To find the electromagnetic-field strengths from this potential, we use our conversion formulae (4.6) and (4.10). Using eqs. (5.13) and (4.6), we find

$$\alpha^{2n+1}_{0,m} = \frac{-iI}{2r}\left(\frac{a}{r}\right)^{2n+2}\delta_{m0}P'_{2n+1}(0) \tag{5.16}$$

for $n = 0, 1, 2, \ldots$. Putting this in (3.2) gives

$$B_r = \frac{Ia}{2r}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n n!}\frac{a^{2n+1}}{r^{2n+2}}P'_{2n+1}(\cos\theta)\,. \tag{5.17}$$

Again, replacing c^{-1} by $(4\pi c)^{-1}$ in eq. (5.46) of ref. [20], we see that (5.17) is the standard expression for B_r in Heaviside-Lorentz units with $c=1$.

To find the angular components of the field strengths, we use eqs. (4.10) and (5.13):

$$\alpha^{2n+1}_{\pm 1,0} = \mp\frac{iI}{2\sqrt{2}}\sqrt{\frac{2n+1}{2n+2}}\frac{a^{2n+2}}{r^{2n+3}}P'_{2n+1}(0)\,, \tag{5.18}$$

which implies that

$$\eta_\pm = \mp\frac{Ia^2}{4\sqrt{2}}\exp[\mp i\varphi_2]\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{1}{r^3}\left(\frac{a}{r}\right)^{2n}P'_{2n+1}(\cos\theta)\,. \tag{5.19}$$

To find the E and B fields, we note that

$$(V_\varphi \pm iV_\theta) = (E_\varphi \mp B_\theta) + i(B_\varphi \pm E_\theta)\,. \tag{5.20}$$

Comparing eqs. (3.1) and (5.19) shows that $B_\varphi = E_\theta = 0$ and thus

$$B_\theta = -\frac{Ia^2}{4}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)!!}{2^n(n+1)!}\frac{1}{r^3}\left(\frac{a}{r}\right)^{2n}P'_{2n+1}(\cos\theta)\,, \tag{5.21}$$

which is just eq. (5.49) of ref. ([20]) expressed in Heaviside-Lorentz units with $c = 1$.

C) Center-fed, linear antenna. As our third basic example we consider the radiation from a thin, linear, centre-fed antenna of length d lying along the z-axis centered at the origin. The current along the antenna vanishes at the end points and is an even fuuction of z. The current density, expressed in spherical co-ordinates, is radial and is confined to lines $\theta = 0$ and $\theta = \pi$. If we assume the current density to vary with frequency ω, as in eq. (2.26), then the spatial part is given by (see ref. ([20]), p. 33)

$$J_r = \begin{cases} \dfrac{I(r)}{2\pi r^2}\,[\delta(\cos\theta - 1) - \delta(\cos\theta + 1)]\,, & r < d/2\,, \\ 0\,, & r \geqslant d/2\,, \end{cases} \tag{5.22}$$

where $I(r)$ is the current, which will be specified later. From the continuity equation the charge density is given by eq. (2.25) with the spatial part given by

$$\varrho = \frac{1}{i\omega}\frac{\mathrm{d}I}{\mathrm{d}r}\left\{\frac{\delta(\cos\theta - 1) - \delta(\cos\theta + 1)}{2\pi r^2}\right\}, \qquad r < d/2\,. \tag{5.23}$$

When these source distributions are expanded in the SU_2 basis, the expansion coefficients are found from eqs. (2.18) and (2.19) to be

$$\varrho^j_{0m} = \begin{cases} \left(\dfrac{2j+1}{2\pi r^2}\right)\dfrac{1}{i\omega}\dfrac{\mathrm{d}I}{\mathrm{d}r}\,\delta_{m0}\,, & j \text{ odd}\,, \\ 0\,, & j \text{ even}\,, \end{cases} \tag{5.24}$$

and

$$J^j_{0m} = \begin{cases} \dfrac{2j+1}{2\pi r^2}\,\delta_{m0}\,I(r)\,, & j \text{ odd}\,, \\ 0\,, & j \text{ even}\,. \end{cases} \tag{5.25}$$

The fact that only terms with $m = 0$ appear reflects the cylindrical symmetry of the source, while only terms with j odd appear due to the behaviour of the source under a parity transformation.

We note in passing that the continuity equation (3.17) requires the expansion coefficients (5.24) and (5.25) to satisfy eq. (3.18), which is easily verified to be the case.

When the observation point has $r > d/2$, we can use eq. (3.19) to find the radial component of the field strengths

$$\alpha^j_{0m} = \frac{i\hat{h}^+_j(\omega r)}{(\omega r)^2}\left(\frac{2j+1}{2\pi}\right)\delta_{n0}\int_0^{d/2}\mathrm{d}r'\left[\frac{\mathrm{d}I}{\mathrm{d}r'}\,\frac{\partial\hat{\jmath}_j(\omega r')}{\partial r'} - \omega^2 I(r')\,\hat{\jmath}_j(\omega r')\right]. \tag{5.26}$$

By noting that

$$\frac{\partial}{\partial r}\left(\hat{\jmath}\,\frac{\mathrm{d}I}{\mathrm{d}r}\right)=\frac{\partial \hat{\jmath}}{\partial r}\frac{\partial I}{\partial r}+\hat{\jmath}\,\frac{\partial^2 I}{\partial r^2}\,,$$

eq. (5.26) can be rewritten as

$$(5.27)\qquad \alpha^j_{0m}=\frac{i\hat{h}^+_j(\omega r)}{\omega r^2}\left(\frac{2j+1}{2\pi}\right)\delta_{m0}\int\limits_0^{d/2}\mathrm{d}r'\left[\frac{\mathrm{d}}{\mathrm{d}r'}\left(\hat{\jmath}_j(\omega r')\,\frac{\mathrm{d}I}{\mathrm{d}r'}\right)-\hat{\jmath}_j(\omega r')\left(\frac{\mathrm{d}^2}{\mathrm{d}r'^2}\,I+\omega^2 I\right)\right].$$

If we now assume that

$$(5.28)\qquad I(z)=I_0\sin\left(\omega d/2-\omega|z|\right)\,,$$

then the second term in eq. (5.27) is zero and the first term can be integrated to give

$$(5.29)\qquad \alpha^j_{0m}=-\,\frac{i\hat{h}^+_j(\omega r)}{\omega r^2}\left(\frac{2j+1}{2\pi}\right)\delta_{m0}\hat{\jmath}_j(\omega d/2)\,I_0\,.$$

We note that this coefficient is $O(r^{-2})$ and so makes no contribution to the time-averaged power in the radiation zone.

For $r>d/2$ the angular coefficients of the field strengths are found from eq. (3.15). In the radiation zone we find

$$(5.30)\qquad \alpha^j_{1,m}=-\,\frac{I_0}{\pi}\,\frac{2j+1}{\sqrt{2j(j+1)}}\,\delta_{n0}\hat{\jmath}_j(\omega d/2)(-\,i)^j\,\frac{\exp\left[i\omega r\right]}{r}$$

and

$$(5.31)\qquad \alpha^j_{-1,m}=0$$

for all j, m. Thus, in the radiation zone, the only nonvanishing appreciable part of the field is

$$(5.32)\qquad \eta_+=-\,\frac{I_0}{\pi}\,\frac{\exp\left[i\omega r\right]}{r}\sum_{\substack{j=1\\(j\ \text{odd})}}^{\infty}\frac{2j+1}{\sqrt{2j(j+1)}}\,(-\,i)^j\hat{\jmath}_j(\omega d/2)\,T^j_{1,0}(u)\,.$$

This expression is in Heaviside-Lorentz units with $c=1$ and can be converted to Gaussian units by replacing $(4\pi)^{-1}$ by c^{-1}.

Since η_0 and η_- are negligible in the radiation zone, it is easy to show that [15]

$$(5.33)\qquad \tfrac{1}{2}|\eta_+|^2=|B_\varphi|^2+|B_\theta|^2\,.$$

This implies that, since $B_r = 0$ to $O(r^{-1})$, the time-averaged power per unit solid angle is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{c}{8\pi}\,|rB|^2 = \frac{c}{16\pi}\,r^2|\eta_+|^2\,. \tag{5.34}$$

Converting eq. (5.32) into Gaussian units, we find

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{I_0^2}{\pi c}\left|\sum_{\substack{j=1\\(j\ \mathrm{odd})}}^{\infty}\frac{2j+1}{\sqrt{2j(j+1)}}(-i)^j\,\mathfrak{j}_j(kd/2)\,T^j_{1,0}\right|^2 \tag{5.35}$$

as the exact result (in the radiation zone).

For a half-wave antenna ($kd = \pi$) the $j = 1$ term of eq. (5.35) is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{I_0^2}{\pi c}\left(\frac{9}{4}\right)\left(\frac{4}{k^2d^2}\right)\left(\frac{\sin^2\theta}{2}\right) = \frac{12I_0^2}{\pi^2 c}\left(\frac{3}{8\pi}\sin^2\theta\right), \tag{5.36}$$

which is exactly the result of ref. [20], p. 765. If we look at the first two terms of eq. (5.35), we find, using the results of appendix B, that

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{I_0^2}{\pi c}\Bigg|\left(-\frac{3i}{2}\right)\mathfrak{j}_1(\pi/2)(-i/\sqrt{2})\sin\theta + \frac{7i}{\sqrt{24}}\,\mathfrak{j}_3(\pi/2)\,\cdot\\ \cdot\left(-\frac{\sqrt{3}i}{4}\sin\theta\right)(5\cos^2\theta-1)\Bigg|^2 = \frac{12I_0^2}{\pi^2c}\left(\frac{3}{8\pi}\right)\sin^2\theta\,\cdot\\ \cdot\left|1-\sqrt{\frac{7}{8}}\,(4.94\cdot10^{-2})(5\cos^2\theta-1)\right|^2,\end{aligned} \tag{5.37}$$

which is exactly eq. (16.123) of ref. [20].

* * *

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Appendix I

The radial helicity vectors.

Helicity basis vectors are normally defined for a wave propagating in the z-direction as

$$\hat{e}_\pm = \frac{1}{\sqrt{2}}\,(\hat{e}_x \pm i e_y)\,. \tag{A.1}$$

The vector $\hat{e}_+$ is called the positive helicity vector since it is an eigenvector of the z-component of angular momentum with eigenvalue $+1$ and, similarly, $\hat{e}_-$ is an eigenvector with eigenvalue -1, and is called the negative helicity vector. These statements are proved by acting on $e_\pm$ expressed in Cartesian co-ordinates with the matrix representing the z-component of angular momentum in Euclidean three-space:

$$e_z\cdot \boldsymbol{S} = S_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{A.2}$$

It is easy to show that

$$S_3\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} = \pm\begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix} \tag{A.3}$$

and thus that

$$S_3\,\hat{e}_\pm = \pm\,\hat{e}_\pm \tag{A.4}$$

and

$$S_3\,\hat{e}_z = 0\hat{e}_z\,. \tag{A.5}$$

If, instead of having a wave propagating along $\hat{e}_z$, we have one propagating along the radial direction, then we can define [16] a radial spin operator $\hat{e}_r\cdot\boldsymbol{S}$, and the vectors

$$\chi_\pm = \frac{1}{\sqrt{2}}(\hat{e}_\theta \pm i\hat{e}_\varphi)\,,\qquad \chi_0 = \hat{e}_r\,. \tag{A.6}$$

It was proved in ref. [16] that

$$(\hat{e}_r\cdot\boldsymbol{S})\,\chi_\pm = \pm\,\chi_\pm \tag{A.7}$$

and

$$(\hat{e}_r\cdot\boldsymbol{S})\,\chi_0 = 0\chi_0\,. \tag{A.8}$$

Thus $\chi_\pm$ are the positive and negative helicity vectors for a wave propagating along e_r and are called the radial helicity vectors.

If we let $\boldsymbol{V}$ be a complex vector field, then we can expand it as

$$\boldsymbol{V} = (\chi_0^*\cdot\boldsymbol{V})\chi_0 + (\chi_+^*\cdot\boldsymbol{V})\chi_+ + (\chi_-^*\cdot\boldsymbol{V})\chi_-\,. \tag{A.9}$$

Using the orthonormality relations

$$\chi_\pm^*\cdot\chi_\mp = \chi_\pm^*\cdot\chi_0 = 0\,,$$

$$\chi_\pm^*\cdot\chi_\pm = \chi_0^*\cdot\chi_0 = 1\,,$$

we find that

$$\chi_+^*\cdot\boldsymbol{V} = iV_+\,,\qquad \chi_-^*\cdot\boldsymbol{V} = -\,iV_-$$

for $V_{\pm}$ defined in eq. (2.1). Thus, expanding $\boldsymbol{V}$ in terms of the radial helicity vectors, we find

$$V = V_r \hat{e}_r + (iV_+)\chi_+ + (-iV_-)\chi_- . \tag{A.10}$$

APPENDIX II

Explicit Expressions for some T^j_{nm}.

From p. 36 of ref. ([21]) the defining relation for T^j_{mn} is

$$T^j_{mn} = (-1)^{2j-m-n}\left[\frac{(j-m)!\,(j+m)!}{(j-n)!\,(j+n)!}\right]^{\frac{1}{2}} \cdot \sum_a \binom{j-n}{a}\binom{j+n}{j-m-a} u_{11}^a\, u_{12}^{j-m-a}\, u_{21}^{j-n-a}\, u_{22}^{m+n+a} , \tag{A.11}$$

where the summation index a runs from max $(0, -m-n)$ to min $(j-m, j-n)$ and

$$\binom{m}{n} = \frac{m!}{(m-n)!\,n!} .$$

The u_{ij} are matrix elements of an element of SU_2

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \exp[i(\varphi_1+\varphi_2)/2] & i\sin\theta/2 \exp[i(\varphi_2-\varphi_1)/2] \\ i\sin\theta \exp[-i(\varphi_2-\varphi_1)/2] & \cos\theta/2 \exp[-i(\varphi_1+\varphi_2)/2] \end{pmatrix} , \tag{A.12}$$

$$\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{A.13}$$

with $0 \leqslant \theta \leqslant \pi$, $0 \leqslant \varphi_1 \leqslant 4\pi$ and $0 \leqslant \varphi_2 \leqslant 2\pi$. Using (A.13) in (A.11) gives

$$T^j_{mn} = (-1)^{2j-m-n}\left[\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}\right]^{\frac{1}{2}} \cdot \sum_a \binom{j-n}{a}\binom{j+n}{j-m-a} \alpha^a \beta^{j-m-a} (-\bar{\beta})^{j-n-a} (\bar{\alpha})^{m+n} . \tag{A.14}$$

If we let

$$P = \alpha^a \beta^{j-m-a} (-\bar{\beta})^{j-n-a} (\bar{\alpha})^{m+n} ,$$

([21]) M. CARMELI and S. MALIN: *Representations of the Rotation and Lorentz Group* (Marcel Dekker, New York, N. Y., 1976).

then a little algebra shows that

$$\text{(A.15)}\qquad P=\left(\frac{\cos\theta+1}{\cos\theta-1}\right)^a\left(\frac{\cos\theta-1}{2}\right)^j\cdot$$
$$\cdot(\operatorname{ctg}\theta/2)^{m+n}\exp[-im\varphi_2]\exp[-in\varphi_1]\,i^{-m-n}\,.$$

Putting (A.15) into (A.14) we find

$$\text{(A.16)}\qquad T^j_{mn}=(-1)^{2j}i^{-m+n}\left[\frac{(j-m)!(j+m)!}{(j-n)!(j+n)!}\right]^{\frac12}\cdot$$
$$\cdot\exp[-im\varphi_2]\exp[-in\varphi_1](\operatorname{ctg}\theta/2)^{m+n}\left(\frac{\cos\theta-1}{2}\right)^j\cdot$$
$$\cdot\sum_a\binom{j-n}{a}\binom{j+n}{j-m-a}\left(\frac{\cos\theta+1}{\cos\theta-1}\right)^a,$$

First, we note that all the φ_1-dependence is contained in an exponential factor, so

$$\text{(A.17)}\qquad T^j_{mn}\propto\exp[-in\varphi_1]$$

and similarly

$$\text{(A.18)}\qquad T^j_{mn}\propto\exp[-im\varphi_2]\,.$$

The θ-dependence is contained in three terms and can be expressed as

$$\text{(A.19)}\qquad \frac{(\operatorname{ctg}\theta/2)^{m+n}}{2}\sum_a\binom{j-n}{a}\binom{j+n}{j-m-a}(\cos\theta-1)^{j-a}(\cos\theta+1)^a\,.$$

On the other hand, the Jacobi polynomial is defined as [22]

$$\text{(A.20)}\qquad P^{(\alpha,\beta)}_\gamma(x)=2^{-\gamma}\sum_{a=0}^{\gamma}\binom{\gamma+a}{a}\binom{\gamma+\beta}{\gamma-a}(x-1)^{\gamma-a}(x+1)^{\gamma+a},$$

so the θ-dependence of T^j_{mn} is essentially that of a Jacobi polynomial, depending on the values of m and n (which determine the limits on the summation in eq. (A.16)). For our purposes it is sufficient to evaluate (A.16) only in some special cases.

Setting $n=0$ and $m=1$ in eq. (A.16), we find

$$\text{(A.21)}\qquad T^j_{10}=(-1)^{2j}i\sqrt{\frac{j+1}{j}}\exp[-i\varphi_2](\operatorname{ctg}\theta/2)\cdot$$
$$\cdot\left(\frac{\cos\theta-1}{2}\right)^j\sum_{a=0}^{j-1}\binom{j}{a}\binom{j}{j-1-a}\left(\frac{\cos\theta+1}{\cos\theta-1}\right)^a.$$

([22]) I. S. GRADSHTEYN and J. M. RYZHIK: *Tables of Integrals, Series and Products* (Academic Press, New York, N. Y., 1965).

Comparison with eq. (A.20) shows that, if we let $\gamma = j - 1$, $\alpha = \beta = 1$, we have

$$T^j_{1,0} = (-1)^{2j}\left(-\frac{i}{2}\right)\sqrt{\frac{j+1}{j}}\exp[-i\varphi_2]\sin\theta\, P^{(1,1)}_{j-1}(\cos\theta)\,. \tag{A.22}$$

But [23]

$$P^{(1,1)}_{j-1}(\cos\theta) = \frac{2}{j+1}\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}P^{(0,0)}_j(\cos\theta) = \frac{2}{j+1}\frac{\mathrm{d}}{\mathrm{d}(\cos\theta)}P_j(\cos\theta)\,,$$

where $P_j(\cos\theta)$ is the usual Legendre polynomial, and thus

$$P^{(1,1)}_{j-1}(\cos\theta) = -\frac{2}{j+1}\frac{1}{\sin\theta}P'_j(\cos\theta)\,. \tag{A.23}$$

This means that

$$T^j_{1,0}(u) = (-1)^{2j}\frac{i}{\sqrt{j(j+1)}}\exp[-i\varphi_2]P'_j(\cos\theta) \tag{A.24}$$

and

$$T^j_{-1,0}(u) = (-1)^{2j}\frac{i}{\sqrt{j(j+1)}}\exp[i\varphi_2]P'_j(\cos\theta)\,. \tag{A.25}$$

In particular,

$$T'_{\pm1,0}(u) = -\frac{i}{\sqrt{2}}\sin\theta\exp[\mp i\varphi_2] \tag{A.26}$$

and

$$T^3_{\pm1,0}(u) = -\frac{\sqrt{3}}{4}\, i\sin\theta(5\cos^2\theta - 1)\exp[\mp i\varphi_2]\,. \tag{A.27}$$

([23]) A. ERDELYI: *Higher Transcendetal Functions*, Vol. **2** (McGraw-Hill, New York, N. Y., 1953), p. 170, 179.

● RIASSUNTO (*)

Si discute il problema della possibilità di avere una formulazione alternata *completa* del l'elettrodinamica classica in termini di una *singola funzione scalare* che può portare anche ad una nuova formulazione dell'elettrodinamica quantistica. Il metodo usa un'analisi armonica e armoniche sferiche pesate secondo lo spin. Si ottengono le equazioni di base per sorgenti che variano nel tempo e le si risolve esplicitamente per tre problemi di base deliberatamente semplici ma caratteristici: cariche puntiformi, anelli di corrente e antenne. Sono stabilite la connessione tra forza di campo e formulazioni di potenziali e la relazione con l'approccio convenzionale alle armoniche sferiche settoriali.

(*) *Traduzione a cura della Redazione.*

Решение основных проблем электродинамики в формулировке с использованием группового пространства.

Резюме (*). — Исследуется вопрос, можно ли получить формулировку классической электродинамики в терминах одной скалярной функции, которая затем позволила бы получить новую формулировку квантовой электродинамики. Предложенный метод использует гармонический анализ и спин-взвешенные сферические гармоники. Мы получаем основные уравнения для зависящих от времени источников. Затем эти уравнения решаются в трех случаях: точечные заряды, петли с током и антенны. Устанавливаются соотношения между напряженностью поля и потенциалом и связь с обычным подходом с векторными сферическими гармониками.

(*) *Переведено редакцией.*

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