IL NUOVO CIMENTO VOL. 97 B, N. 1 11 Gennaio 1987

Expectation Values in the Aharonov-Bohm Effect.

M. D. SEMON

Department of Physics, Bates College - Lewiston, ME 04240

J. R. TAYLOR

Department of Physics, University of Colorado - Boulder, CO 80309

(manoscritto revisionato ricevuto il 17 Giugno 1986)

Summary. — It has been well established that, as predicted by Aharonov and Bohm, electron interference patterns can be shifted by the introduction of electromagnetic potentials, even if the electrons never enter the region in which the fields are nonzero. In this paper we prove that, even though the interference pattern shifts, none of the moments of the electron's position $\boldsymbol{r}$, nor of its kinetic momentum $\boldsymbol{\pi}$, are affected. On the other hand, we prove that the expectation value of the operator $\sin \boldsymbol{a}\cdot\boldsymbol{\pi}$ (with $\boldsymbol{a}$ a certain fixed vector), which was first introduced by Aharonov, Pendleton and Peterson, *does* shift.

PACS. 03.65. – Quantum theory; quantum mechanics.

1. – Introduction.

More than 25 years have passed since AHARONOV and BOHM predicted the remarkable phenomenon that has come to be called the Aharonov-Bohm effect (1). Their idea ran so counter to prevailing beliefs about electromagnetic potentials that it met with considerable skepticism, and, even though the effect was observed experimentally, there were several attempts to interpret it in a more traditional way. Today, however, both the Aharonov-Bohm prediction and its experimental verification are generally accepted as beyond question.

(1) Y. AHARONOV and D. BOHM: *Phys. Rev.*, **115**, 485 (1959).

In this paper we wish to draw attention to a somewhat surprising aspect of the Aharonov-Bohm effect that has hitherto gone almost unnoticed. We certainly make no claim that our result is as surprising as the original prediction of Aharonov and Bohm, but it does serve to reemphasize the remarkable nature of this fascinating experiment.

The Aharonov-Bohm, or AB, experiment can be arranged in several different ways, of which the simplest to discuss is that proposed by AHARONOV and BOHM themselves. A beam of electrons is directed at a barrier with two identical slits. The resulting interference pattern, consisting of regularly spaced two-slit fringes underneath the one-slit envelope, is observed on a screen far beyond the barrier. The two-slit pattern is now caused to shift underneath the fixed one-slit envelope by the introduction of magnetic or electric potentials, $\boldsymbol{A}$ or φ. The main point of the AB effect is that one can arrange conditions so that, although the electrons experience nonzero potentials, they never enter the region in which the fields $\boldsymbol{B} = \nabla \times \boldsymbol{A}$ or $\boldsymbol{E} = -\nabla\varphi$ are nonzero. Even under these conditions the electron interference pattern *does* shift.

The AB effect is, of course, a quantum-mechanical effect. In classical mechanics a charged particle responds only to electromagnetic *fields* (not directly to the *potentials*) and, if the particle never enters the region in which the fields are nonzero, then its motion cannot be affected in any way by the presence or absence of the fields. In quantum mechanics it is the potentials $\boldsymbol{A}$ or φ that appear in the Schrödinger equation. As AHARONOV and BOHM made clear, the presence of nonzero potentials can shift the interference pattern even when the electrons never enter the region in which the fields $\boldsymbol{B}$ or $\boldsymbol{E}$ are nonzero.

In this paper we prove that, although the electron interference pattern shifts as predicted by AHARONOV and BOHM, none of the moments of the electron's position or momentum are affected. Specifically, let C denote the operator

$$C = x^l y^m z^n \pi_x^p \pi_y^q \pi_z^r \,. \tag{1}$$

In this definition l, m, n, p, q and r are arbitrary nonnegative integers, $\boldsymbol{r} = (x, y, z)$ is the position operator and $\boldsymbol{\pi} = (\pi_x, \pi_y, \pi_z)$ is the kinetic momentum,

$$\boldsymbol{\pi} = m\boldsymbol{v} = \boldsymbol{p} - q\boldsymbol{A} \,. \tag{2}$$

Here $\boldsymbol{p}$ is the canonical momentum, represented by $-i\nabla$, and q is the particle's charge ($q = -e$ for an electron). With these definitions we prove that, in both the magnetic and electric AB experiments, the expectation value

$$\langle C \rangle_t = \langle \psi(\boldsymbol{r}, t),\ C\psi(\boldsymbol{r}, t)\rangle \tag{3}$$

is independent, at all times t, of the potentials $\boldsymbol{A}$ or φ, that is, $\langle C \rangle_t$ has the same value in the presence of the potentials $\boldsymbol{A}$ or φ as it has when $\boldsymbol{A}$ and φ are identically zero. This in turn implies that the expectation value of any polynomial in the six variables $\boldsymbol{r}$ and $\boldsymbol{\pi}$ is independent of the magnetic or electric potentials.

In sect. **2** we review briefly the AB effect and the proof that the interference pattern shifts when an appropriate magnetic or electric potential is switched on. This allows us to introduce our notation and assumptions on the wave functions. In sect. **3** we prove our « no-shift theorem » that the expectation value of any polynomial in the variables $\boldsymbol{r}$ and $\boldsymbol{\pi}$ does not shift when the potential is introduced.

Finally in sect. **4** we discuss an operator

$$D = \sin \boldsymbol{a}\cdot\boldsymbol{\pi}\,, \tag{4}$$

where $\boldsymbol{a}$ is a fixed vector pointing in the direction from one slit to the other. This operator was introduced by AHARONOV, PENDLETON and PETERSON (2) (APP) and called by them the modular momentum. We shall prove that—as suggested by APP—the expectation value of D *does* change when magnetic or electric potentials are introduced. We conclude sect. **4** with a brief discussion of the seeming paradox that, while no powers of $\boldsymbol{\pi}$ are shifted, the function $\sin \boldsymbol{a}\cdot\boldsymbol{\pi}$ *is* shifted; in particular, we describe why this is not a contradiction.

To conclude this introduction we most mention that some particular cases of the no-shift theorem have been given previously. The paper of APP[2] gives the no-shift theorem for powers of the momentum, mainly in the context of the electric AB effect. (In the case of the electric effect there is no distinction between the canonical momentum $\boldsymbol{p}$ and the kinetic momentum $\boldsymbol{\pi}$, of course.) In the recent review of Olariu and Popescu (3) the no-shift theorem for $\boldsymbol{r}$, $\boldsymbol{\pi}$, and $\boldsymbol{\pi}^2$ is proved for the magnetic effect. KOBE (4) has given an example of wave functions that appear to violate the no-shift theorem, but, as emphasized by OLARIU and POPESCU, Kobe's assumptions violate an essential requirement of any realistic version of the AB experiment—namely that the two slits cannot overlap one another. We ourselves were led to the no-shift theorems by the paper of AAP[2] and have already emphasized their importance at the 1984 symposium on quantum mechanics at SUNY-Albany (5).

Finally, we should emphasize that, in the case of $\boldsymbol{\pi}$ at least, the no-shift theorem is easy to understand: the evolution of $\langle\boldsymbol{\pi}\rangle_t$ is given by Ehrenfest's theorem in terms of expectation values of $\boldsymbol{v}\times\boldsymbol{B}$ or $\boldsymbol{E}$, and in the AB experiment

(2) Y. AHARONOV, H. PENDLETON and A. PETERSON: *Int. J. Theor. Phys.*, **2**, 213 (1969).

(3) S. OLARIU and I. I. POPESCU: *Rev. Mod. Phys.*, **57**, 339 (1985).

(4) D. H. KOBE: *Ann. Phys.* (*N. Y.*), **123**, 381 (1979).

(5) M. SEMON and J. R. TAYLOR: in *Fundamental Questions in Quantum Mechanics*, edited by A. INOMATA and L. ROTH (Gordon and Breach, New York, N. Y., 1986), p. 191.

these expectation values are zero. Therefore, the evolution of $\langle\boldsymbol{\pi}\rangle_t$ is unaffected by the presence or absence of the fields. This simple and transparent argument can be extended by induction to include higher powers of the momentum and is, in fact, the argument sketched by APP in the electric case. However, it does not seem to extend to arbitrary powers of the position operator. We present here a method of proof that applies equally to the no-shift theorems for powers of $\boldsymbol{r}$ and $\boldsymbol{\pi}$ and to the shift of the modular momentum (4), in both electric and magnetic AB effects.

2. – Notation and review.

We consider a beam of charged particles traveling in the positive z-direction towards a two-slit barrier lying in the plane $z = 0$. The exact shape of the slits does not matter, but they must be identical, one being the result of translating the other through a displacement $\boldsymbol{d}$ in the x-direction. It is essential that the two slits do not overlap; in the magnetic AB effect, for example, this allows us to place a thin solenoid in the shadow of the barrier, between the two slits.

We assume, as usual, that in the region beyond the barrier ($z > 0$) the actual wave packet $\psi(\boldsymbol{r}, t)$ can be small approximated as the sum,

$$\psi(\boldsymbol{r}, t) = \psi_1(\boldsymbol{r}, t) + \psi_2(\boldsymbol{r}, t)\,, \tag{5}$$

where $\psi_\nu(\boldsymbol{r}, t)$, with $\nu = 1$ or 2, is the wave packet that would have been transmitted if the ν-th slit were open and the other closed [6]. We assume that at a certain time $t = 0$, just after the wave passes the barrier, the packets ψ_1 and ψ_2 are nonoverlapping; that is, the supports of $\psi_1(\boldsymbol{r}, 0)$ and $\psi_2(\boldsymbol{r}, 0)$ are disjoint. (This assumption also is only approximate, but is certainly a very good approximation.) Naturally, as time passes, the packets ψ_1 and ψ_2 do overlap again, allowing us to observe interference between them. We assume further that the incident wave packet is wide enough compared to the slit separation that $\psi_2(\boldsymbol{r}, t)$ is just the translation through $\boldsymbol{d}$ of $\psi_1(\boldsymbol{r}, t)$:

$$\psi_2(\boldsymbol{r}, t) = \psi_1(\boldsymbol{r} - \boldsymbol{d}, t) = \exp\left[-i\boldsymbol{d}\cdot\boldsymbol{p}\right]\psi_1(\boldsymbol{r}, t)\,. \tag{6}$$

---

(6) Since elementary discussions of the two-slit experiment often imply that (5) follows from the superposition principle, it is perhaps worth emphasizing that (5) is only an approximation. The point is that ψ_1 and ψ_2 satisfy different boundary conditions at the barrier (ψ_1 is zero with discontinuous derivative at slit 2 and *vice versa*). However, to the extent that ψ_1 is very close to zero in a whole neighborhood of slit 2, and *vice versa*, (5) is an excellent approximation.

If the incident wave (in the region $z < 0$) is well peaked in momentum with mean momentum p_0 and reduced wave-length $\lambda = 1/p_0$, then, at points far from the slits in $z > 0$, eq. (6) implies that ψ_2 differs from ψ_1 by the phase factor (7) $\exp[-i\hat{r}\cdot\boldsymbol{d}/\lambda]$. Thus the total intensity $|\psi|^2$ far beyond the slits is

$$(7)\qquad |\psi(\boldsymbol{r}, t)|^2 = |\psi_1(\boldsymbol{r}, t) + \psi_2(\boldsymbol{r}, t)|^2 =$$
$$= |1 + \exp[i\hat{r}\cdot\boldsymbol{d}/\lambda]|^2 |\psi_1(\boldsymbol{r}, t)|^2 = [4\cos^2(\hat{r}\cdot\boldsymbol{d}/2\lambda)]|\psi_1(\boldsymbol{r}, t)|^2 .$$

Here the first factor defines the two-slit interference pattern and the second, $|\psi_1(\boldsymbol{r}, t)|^2$, is the one-slit envelope. In a two-dimensional situation (the infinitely long slits of many elementary discussions) the two-slit factor reduces to

$$4\cos^2(\hat{r}\cdot\boldsymbol{d}/2\lambda) = 4\cos^2[d(\sin\theta)/2\lambda]$$

with maxima given by the familiar condition $d(\sin\theta)/2\lambda = n\pi$ or

$$d\sin\theta = n\lambda .$$

Magnetic AB *effect*. – In the magnetic Aharonov-Bohm experiment the standard two-slit experiment is modified by the introduction of a magnetic field which is arranged so that the electrons never enter the region of nonzero field. This is accomplished by placing a narrow solenoid just beyond the barrier, between the two slits, as shown in fig. 1. The solenoid carries a constant current

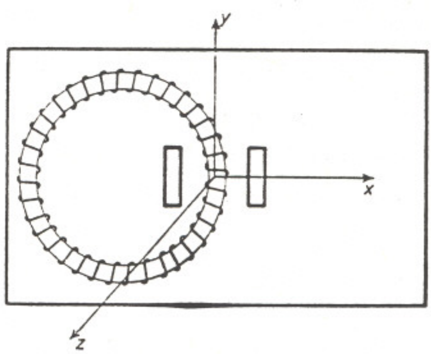

Fig. 1. – The magnetic AB effect. The electrons propagate in the z-direction, coming from the region $z < 0$, and strike the two-slit barrier located in the (x, y)-plane. The solenoid is placed just downstream of the barrier, between the two slits.

(7) To prove this, expand ψ_1 in terms of stationary states, whose asymptotic form is $\exp[ip_0 r]/r$. The corresponding expansion of ψ_2 has r replaced by $|\boldsymbol{r} - \boldsymbol{d}|$, which for large r is approximately $r - \hat{\boldsymbol{r}}\cdot\boldsymbol{d}$. Thus ψ_2 differs from ψ_1 by the phase factor $\exp[-ip_0\hat{\boldsymbol{r}}\cdot\boldsymbol{d}]$.

and produces a constant magnetic field B. The exact shape of the solenoid does not matter, but it is convenient to consider a closed solenoid, such as the torus shown in fig. 1, since this lets us assume that the B field is confined entirely inside the solenoid ([8]). Therefore, since the solenoid is placed immediately in the shadow of the barrier, the electrons do not enter the region in which B is nonzero ([9]).

On the other hand, the magnetic potential $\boldsymbol{A}$ is *not* zero outside the solenoid and, as pointed out by AHARONOV and BOHM, its effect is to change the relative phase of the two waves ψ_1 and ψ_2 and hence to shift the observed two-slit interference pattern. This can be (and has been) proved in several ways. We shall prove it again now, because we shall use the same technique to prove our main results in sect. **3** and **4**. Our proof uses the following theorem:

Theorem. Let R denote any simply connected region, in which the magnetic field is zero: $\nabla\times\boldsymbol{A}=0$. Let $\alpha(\boldsymbol{r})$ be the single-valued function

$$\alpha(\boldsymbol{r})=q\int_{\boldsymbol{r}_0}^{\boldsymbol{r}}\boldsymbol{A}(\boldsymbol{r}')\cdot\mathrm{d}\boldsymbol{r}'\,, \tag{8}$$

where $\boldsymbol{r}_0$ is any fixed point and the path of integration lies entirely in R. (Notice that $\alpha(\boldsymbol{r})$ is single-valued because $\nabla\times\boldsymbol{A}=0$ and R is simply connected.) Finally, let $\psi(\boldsymbol{r})$ be any wave function that vanishes outside R and let $\boldsymbol{\pi}$ be the kinetic momentum

$$\boldsymbol{\pi}=\boldsymbol{p}-q\boldsymbol{A}\,,$$

where

$$\boldsymbol{p}=-i\nabla$$

is the canonical momentum. Then, for any component $i=1,2,3$ of $\boldsymbol{\pi}$ and $\boldsymbol{p}$ and for any nonnegative integer n,

$$\pi_i^n\exp[i\alpha(\boldsymbol{r})]\,\psi(\boldsymbol{r})=\exp[i\alpha(\boldsymbol{r})]\,p_i^n\,\psi(\boldsymbol{r})\,. \tag{9}$$

We can describe relation (9) as an intertwining relation for the operators π_i and p_i and the phase factor $\exp[i\alpha(\boldsymbol{r})]$. The content of this theorem is

---

([8]) A simpler arrangement is, perhaps, to consider an infinitely long, straight solenoid, but this causes some ambiguity associated with the return field on the outside.

([9]) This claim has been tested experimentally by TONOMURA, who did the AB experiment both with and without a gold shield around the source of magnetic field. That the results were the same in both cases (and agreed with the AB predictions) verified that the electrons do not penetrate the region of nonzero field and that the AB effect does not depend on such a penetration. This beautiful experiment is described in the conference proceedings of ref. ([5]).

certainly well known and is implicit in many standard works ([1,10]). On the other hand, we have not seen it stated explicitly in this simple and rather elegant form. Its proof is straightforward and depends on the obvious fact that

$$\nabla\alpha(\boldsymbol{r}) = q\boldsymbol{A}(\boldsymbol{r})\,.$$

Result (9) cannot be applied directly to the complete wave function ψ of the two-slit experiment, since the region defined by a barrier with two slits is not simply connected. It can, however, be applied to the functions, ψ_1 and ψ_2, appropriate to the separate single slits. We add a superscript 0 to identify the wave functions for zero magnetic field. These wave functions satisfy the time-dependent Schrödinger equation

$$\left(\frac{p^2}{2m} + V\right)\psi_\nu^0(\boldsymbol{r}, t) = i\,\frac{\partial\psi_\nu^0}{\partial t}(\boldsymbol{r}, t)\,, \tag{10}$$

where ν identifies the open slit ($\nu = 1$ or 2) and V is the potential describing the barrier. (In what follows, we take for granted that, after the wave has passed the barrier, the effect of V is negligible and ψ_ν^0 evolves according to the free Hamiltonian $p^2/2m$.) We now define two phase functions $\alpha_\nu(\boldsymbol{r})$ by eq. (8) with the path of integration from any $\boldsymbol{r}_0$ in $z < 0$ through the ν-th slit to any point $\boldsymbol{r}$ in $z > 0$. (Note that $\alpha_\nu(\boldsymbol{r})$ depends on $\boldsymbol{r}$, but is independent of t since $\boldsymbol{A}(\boldsymbol{r})$ is.) If we now define

$$\psi_\nu(\boldsymbol{r}, t) = \exp\left[i\alpha_\nu(\boldsymbol{r})\right]\psi_\nu^0(\boldsymbol{r}, t)\,, \tag{11}$$

then it follows from (9) and (10) that ψ_ν satisfies

$$\left(\frac{\pi^2}{2m} + V\right)\psi_\nu(\boldsymbol{r}, t) = i\,\frac{\partial\psi_\nu}{\partial t}(\boldsymbol{r}, t)\,. \tag{12}$$

This is the time-dependent Schrödinger equation for the electron in the presence of the magnetic potential $\boldsymbol{A}(\boldsymbol{r})$. When $t \to -\infty$, the wave function ψ_ν^0 represents a wave packet approaching the barrier from afar and the same is therefore true of ψ_ν. Therefore, ψ_ν is *the* wave function for a single slit in the presence of the constant magnetic potential $\boldsymbol{A}(\boldsymbol{r})$.

It follows from (11) that the complete wave function, in the presence of the magnetic potential, is

$$\psi(\boldsymbol{r}, t) = \psi_1(\boldsymbol{r}, t) + \psi_2(\boldsymbol{r}, t) = \exp\left[i\alpha_1(\boldsymbol{r})\right]\psi_1^0(\boldsymbol{r}, t) + \exp\left[i\alpha_2(\boldsymbol{r})\right]\psi_2^0(\boldsymbol{r}, t)\,. \tag{13}$$

([10]) J. J. SAKURAI: *Advanced Quantum Mechanics* (Addison-Wesley, Reading, Mass., 1969), p. 15.

The effect of the magnetic potential is simply to change the phases of the two components, ψ_1 and ψ_2, by the different, $\boldsymbol{r}$-dependent phases $\alpha_1(\boldsymbol{r})$ and $\alpha_2(\boldsymbol{r})$. The change in the *relative* phase of $\psi_1(\boldsymbol{r})$ and $\psi_2(\boldsymbol{r})$ (at any $\boldsymbol{r}$ in $z>0$) is independent of $\boldsymbol{r}$, since it is just the fixed number

$$\alpha = \alpha_1(\boldsymbol{r}) - \alpha_2(\boldsymbol{r}) = q\oint \boldsymbol{A}\cdot \mathrm{d}\boldsymbol{r}\,, \tag{14}$$

where the closed path of the integral goes through the first slit and returns through the second. This integral is just the total magnetic flux Φ along the solenoid. Therefore the magnetic potential simply moves the whole two-slit pattern—the first factor in eq. (7)—sideways by the phase $\alpha = q\Phi$, without affecting the one-slit envelope $|\psi_1(\boldsymbol{r}, t)|^2$ at all.

Electric AB *effect.* While the magnetic AB effect has been repeatedly verified experimentally, the same is not true of the electric effect [11]. Further, the electric effect is somewhat less interesting theoretically since the distinction between the kinetic and canonical momenta ($\boldsymbol{\pi}$ and $\boldsymbol{p}$) disappears when there is no magnetic field. For both of these reasons we shall focus mainly on the magnetic effect here. Nonetheless, we describe the electric effect briefly since all our results apply to it as well.

In the electric effect, a potential difference is established between the components ψ_1 and ψ_2 of the wave by placing two conducting cylinders just beyond the slits, as shown in fig. 2. A potential difference is applied between the cylinders

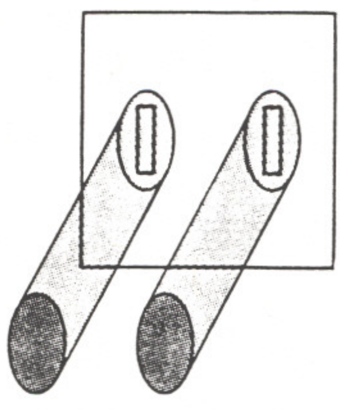

Fig. 2. – In the electric AB effect a potential difference is applied between two conducting cylinders placed just beyond the two slits.

[11] A recent report of an experimental verification is G. MATTEUCI and G. POZZI: *Phys. Rev. Lett.*, **54**, 2469 (1985); see also S. OLARIU and I. I. POPESCU: *Rev. Mod. Phys.*, **57**, 412 (1985).

after the wave packet has entered them (at $t = 0$) and is switched off again before they leave (at time $t = T$). (This would obviously be extremely hard in practice, but is nonetheless possible in principle.) To the extent that the potential is constant inside each cylinder, the electron experiences no electric field. On the other hand, the electric potential φ modifies the wave function so that

$$\psi_\nu(\boldsymbol{r}, t) = \exp[i\alpha_\nu(t)]\,\psi_\nu^0(\boldsymbol{r}, t)\,, \tag{15}$$

where, as before, ψ_ν^0 denotes the wave function for the ν-th slit, but

$$\alpha_\nu(t) = q\int_0^t \varphi_\nu(t')\,\mathrm{d}t'\,, \tag{16}$$

where $\varphi_\nu(t)$ is the electric potential inside the ν-th cylinder. That ψ_ν, as defined in (15), satisfies the Schrödinger equation with electric potential $\varphi_\nu(t)$ is easily verified by direct differentiation. The result of the applied potential difference is that the relative phase of the two emerging wave packets is shifted by the amount

$$\alpha = q\int_0^T [\varphi_1(t) - \varphi_2(t)]\,\mathrm{d}t\,.$$

Just as in the magnetic AB effect, this implies a shift of the two-slit pattern underneath the fixed one-slit envelope.

Assumptions on the wave functions. In the next two sections we prove our two theorems. Naturally, these depend on the assumptions that went into our model of the AB experiment, and we therefore conclude this section by reiterating what those assumptions are. First, we take for granted that $\psi = \psi_1 + \psi_2$ in the region $z > 0$, downstream from the two-slit barrier. Next we assume that, at a certain time $t = 0$, just after the wave packet passes through the slits, the packets ψ_1 and ψ_2 do not overlap. Finally, we assume that, for $t > 0$, the evolution of ψ_1 and ψ_2 is unaffected by the barrier; that is the term V in eqs. (10) and (12) can be neglected for $t > 0$.

With these assumptions our model of the AB experiment is analogous to the model usually used to discuss two-slit experiments in optics. In both cases one approximates the net effect of the complicated interation between the slits and the incident beam by a wave function, $\psi = \psi_1 + \psi_2$, satisfying the assumptions above. That this is a realistic model is confirmed by the many experiments that have verified the interference patterns that it predicts. (See ref. ([3]), pp. 390-408, for a comprehensive survey of the experiments with electrons, and p. 360 for a critique of various models.)

3. – Proof of the no-shift theorem.

We consider first the magnetic AB effect since the extension of our arguments to the electric effect will prove entirely straightforward. Similarly, we examine first just powers of any one component of $\boldsymbol{r}$ or $\boldsymbol{\pi}$, since the extension to operators of the form (1), involving products of powers of all six components of $\boldsymbol{r}$ and $\boldsymbol{\pi}$, is likewise trivial.

We begin with the operator π_i^n, with $i = 1, 2$, or 3, and n an arbitrary nonnegative integer. We wish to show that the expectation value

$$\langle\pi_i^n\rangle_t = \langle\psi(\boldsymbol{r}, t), \pi_i^n\,\psi(\boldsymbol{r}, t)\rangle \tag{17}$$

is independent of the presence of the magnetic potential $\boldsymbol{A}$. Substituting eq. (10), $\psi = \sum_\nu \exp[i\alpha_\nu(\boldsymbol{r})]\,\psi_\nu^0$, we find that

$$\langle\pi_i^n\rangle_t = \sum_{\nu=1}^{2}\sum_{\mu=1}^{2}\langle\exp[i\alpha_\nu(\boldsymbol{r})]\,\psi_\nu^0(\boldsymbol{r}, t), \pi_i^n \exp[i\alpha_\mu(\boldsymbol{r})]\,\psi_\mu^0(\boldsymbol{r}, t)\rangle\,.$$

Using theorem (9), we can move π_i^n through the phase factor on its r.h.s. to give

$$\langle\pi_i^n\rangle_t = \sum_\nu\sum_\mu\langle\psi_\nu^0(\boldsymbol{r}, t), \exp i[\alpha_\mu(\boldsymbol{r}) - \alpha_\nu(\boldsymbol{r})]\,p_i^n\,\psi_\mu^0(\boldsymbol{r}, t)\rangle\,.$$

This double sum contains four terms in all. In the two terms with $\nu = \mu$, the phase factor is equal to unity; in the other two, it is the constant $\exp[\pm i\alpha]$. In either case the phase factor is a constant and can be taken outside the integral to give

$$\langle\pi_i^n\rangle_t = \sum_{\nu=1}^{2}\langle\psi_\nu^0(\boldsymbol{r}, t), p_i^n\,\psi_\nu^0(\boldsymbol{r}, t)\rangle + 2\,\mathrm{Re}\exp[i\alpha]\,\langle\psi_2^0(\boldsymbol{r}, t), p_i^n\,\psi_1^0(\boldsymbol{r}, t)\rangle\,. \tag{18}$$

The two wave packets ψ_1^0 and ψ_2^0 in (18) evolve according to the free Hamiltonian $p^2/2m$ when $t \geqslant 0$. Therefore, the matrix elements of p_i^n are independent of time and can be replaced by their values at $t = 0$:

$$\langle\pi_i^n\rangle_t = \sum_{\nu=1}^{2}\langle\psi_\nu^0(\boldsymbol{r}, 0), p_i^n\,\psi_\nu^0(\boldsymbol{r}, 0)\rangle + 2\,\mathrm{Re}\exp[i\alpha]\,\langle\psi_2^0(\boldsymbol{r}, 0), p_i^n\,\psi_1^0(\boldsymbol{r}, 0)\rangle\,. \tag{19}$$

Now, at time $t = 0$, the wave packets ψ_1^0 and ψ_2^0 are nonoverlapping. Since $p_i^n = (-i\,\partial/\partial r_i)^n$, it follows that the final matrix element in (19) is zero and we conclude that

$$\langle\pi_i^n\rangle_t = \sum_{\nu=1}^{2}\langle\psi_\nu^0(\boldsymbol{r}, 0), p_i^n\,\psi_\nu^0(\boldsymbol{r}, 0)\rangle\,. \tag{20}$$

This is independent of the magnetic field and equal to its value for the case that the field is switched off. This completes the proof for the operator π_i^n.

We consider next the expectation value [12]

$$\langle r_i^n \rangle_t = \langle \psi(\boldsymbol{r}, t), r_i^n \psi(\boldsymbol{r}, t)\rangle .$$

We can make the same substitutions as before, and, since r_i commutes with $\exp[i\alpha_\mu(\boldsymbol{r})]$, we arrive at the equation corresponding to (18):

$$(21)\qquad \langle r_i^n \rangle_t = \sum_{\nu=1}^{2} \langle \psi_\nu^0(\boldsymbol{r}, t), r_i^n \psi_\nu^0(\boldsymbol{r}, t)\rangle + 2 \operatorname{Re} \exp[i\alpha] \langle \psi_2^0(\boldsymbol{r}, t), r_i^n \psi_2^0(\boldsymbol{r}, t)\rangle .$$

Since the two wave functions evolve freely, we can rewrite the final matrix element as

$$\langle \psi_2^0(\boldsymbol{r}, t), r_i^n \psi_1^0(\boldsymbol{r}, t)\rangle = \left\langle \psi_2^0(\boldsymbol{r}, 0), \left(r_i - \frac{p_i t}{m}\right)^n \psi_1^0(\boldsymbol{r}, 0)\right\rangle =$$

$$= \left\langle \psi_2^0(\boldsymbol{r}, 0), \left(r_i - \frac{it}{m}\frac{\partial}{\partial r_i}\right)^n \psi_1^0(\boldsymbol{r}, 0)\right\rangle .$$

which is zero since the wave functions ψ_1^0 and ψ_2^0 do not overlap at $t = 0$. Thus, according to (21), the expectation value $\langle r_i^n \rangle_t$ is equal to the sum of two terms, neither of which depends on the magnetic potential.

This completes the proof of the no-shift theorem for the operator r_i^n. It should be clear that the arguments used for π_i^n and r_i^n can be combined to cover any product of components of $\boldsymbol{\pi}$ and $\boldsymbol{r}$ and our proof is complete for the magnetic AB effect.

The proof for the electric effect follows the same steps, but is much simpler. The kinetic and canonical momenta aree qual ($\boldsymbol{\pi} = \boldsymbol{p}$), and the phase factors $\exp[i\alpha_\mu(\boldsymbol{r})]$ of the magnetic case are replaced by factors $\exp[i\alpha_\mu(t)]$ that are independent of $\boldsymbol{r}$ and so commute with both $\boldsymbol{r}$ and $\boldsymbol{p}$. With these two simplifications the proof goes through exactly as in the magnetic case.

One might feel that, if none of the moments of the electron distribution shift, then the distribution itself could not shift. This feeling is based on the conviction that any « reasonable » distribution is uniquely determined by its moments and, if this were true, then our no-shift theorem would indeed contradict the observed shift of the interference pattern in the AB experiment.

It turns out that the conditions under which a distribution is uniquely determined by its moments are rather complicated [13]. We can, nevertheless,

---

(12) We assume that the moment in question is finite. If it is not, then the no-shift theorem is still true but is less interesting.

(13) See, for example, H. CRAMER: *Methamatical Methods of Statistics* (Princeton University Press, Princeton, N. J., 1946), p. 176; or W. FELLER: *An Introduction to Probability Theory and Its Applications*, Vol. 2, 2nd edition (John Wiley and Sons, New York, N. Y., 1971), p. 227.

make three simple observations. First, one simple sufficient condition is that the distribution has compact support, but our wave function certainly does not satisfy this condition. (Although our assumptions imply that ψ_1 and ψ_2 have compact support at time $t = 0$, at this time the packets do not overlap and there is as yet no interference pattern to shift. By the time the packets do interfere they certainly no longer have compact support.) Second, a necessary condition that a distribution be determined by its moments is that all of its moments be finite; and we certainly have no guarantee that all moments of $|\psi(\boldsymbol{r}, t)|^2$ are finite. Third, there is one clear, general conclusion that we can draw: we know that the distribution $|\psi(\boldsymbol{r}, t)|^2$ does shift in the AB effect and we have proved that none of its moments do shift. Therefore, it is clear that $|\psi(\boldsymbol{r}, t)|^2$ is not in the class of distributions that are determined uniquely by their moments.

Finally the no-shift theorem is sufficiently surprising that it would be nice to find a solvable model in which one could see explicitly that the moments do not shift. Unfortunately, there are very few solvable models and all of them involve plane waves, which lead to moments that are divergent. Nevertheless, we are preparing a second paper, in which we shall present two solvable models in which one can see that those moments which are finite do not shift ([14]).

4. – Modular momentum.

In this final section we examine briefly an operator introduced by APP ([2]) and called by them the modular momentum. We first sketch the motivation for considering this operator.

Having established that none of the moments of either $\boldsymbol{r}$ or $\boldsymbol{\pi}$ are changed in the AB effect, one is naturally led to ask whether there are any operators that *do* change. The answer to this question is certainly « yes ». If ψ and ψ^0 are the wave functions with and without the AB field, then we have seen that ψ and ψ^0 are linearly independent, and this immediately guarantees the existence of self-adjoint operators D for which

$$\langle\psi^0, D\psi^0\rangle \neq \langle\psi, D\psi\rangle . \tag{22}$$

([14]) One solvable model that seems at first sight to contradict our theorem is the scattering by a magnetic « string », originally analysed by AHARONOV and BOHM in ref. ([1]). This leads to a nonzero cross-section that is symmetric about the forward direction and would seem to shift any even moment of x and π_x. We remark first that the geometry of this experiment (which has no two-slit barrier) does not fit the conditions of our theorem. More important, OLARIU and POPESCU (ref. ([3])) have analysed this experiment in terms of wave packets and shown that it is in fact consistent with our results.

It is actually quite easy to construct examples of operators satsifying eq. (22). For example, let V denote a volume in position space and let $P(V)$ be the projection operator onto this volume ([15]). The expectation value $\langle\psi, P(V)\psi\rangle$ is just the probability of finding the electron in the volume V and, with V suitably chosen, this probability is certainly different for ψ and ψ^0. OLARIU and POPESCU ([3]) have pointed out that parity is another operator satisfying (22), since the introduction of an external magnetic field certainly changes the parity.

Nevertheless it is natural to inquire whether one could find an operator whose expectation value shifts and which has a more obvious dynamical significance. It was in this spirit that APP introduced their « modular momentum », defined as

$$D = \sin \boldsymbol{a}\cdot\boldsymbol{\pi} = \sin a\pi_x\,, \tag{23}$$

where $\boldsymbol{a}$ is a vector pointing in the x direction (that is, from slit 1 to slit 2). They argued (in the context of an AB effect using a diffraction grating) that this is a natural operator to consider and that its study enhances our understanding of the dynamics of the AB experiment ([16]). Here we wish to add some weight to these claims by proving that, in both the magnetic and electric two-slit experiments, the expectation value of the operator (23) does indeed shift—even though none of the moments of π_x do. We shall consider explicitly just the magnetic AB effect since, just as in sect. **3**, our arguments apply equally to the electric effect but are simpler, since $\boldsymbol{\pi} = \boldsymbol{p}$ in that case.

We can expand $\sin a\pi_x$ as

$$\sin a\pi_x = \big(\exp[ia\pi_x] - \exp[-ia\pi_x]\big)/2i$$

and, since it is more convenient to examine the separate exponentials, we consider first

$$\langle\exp[ia\pi_x]\rangle_t = \langle\psi(\boldsymbol{r}, t), \exp[ia\pi_x]\,\psi(\boldsymbol{r}, t)\rangle\,.$$

By (13), this is equal to

$$\langle\exp[ia\pi_x]\rangle_t = \sum_\nu\sum_\mu \langle\exp[i\alpha_\nu(\boldsymbol{r})]\,\psi^0_\nu(\boldsymbol{r}, t), \exp[ia\pi_x]\exp[i\alpha_\mu(\boldsymbol{r})]\,\psi^0_\mu(\boldsymbol{r}, t)\rangle\,.$$

Using theorem (9), we can move $\exp[ia\pi_x]$ through the phase factor and replace it by $\exp[iap_x]$:

$$\langle\exp[ia\pi_x]\rangle_t = \sum_\nu\sum_\mu \exp[i(\alpha_\mu - \alpha_\nu)]\,\langle\psi^0_\nu(\boldsymbol{r}, t), \exp[iap_x]\,\psi^0_\mu(\boldsymbol{r}, t)\rangle\,. \tag{24}$$

([15]) That is, $P(V)\,\psi(\boldsymbol{r}) = \psi(\boldsymbol{r})$ if $\boldsymbol{r}$ is in V, but $P(V)\,\psi(\boldsymbol{r}) = 0$ otherwise.

([16]) For a review of their arguments in the context of the standard AB effect, see ref. ([5]).

(The remaining phase factor has been taken outside the integral since it is independent of $\boldsymbol{r}$.) Since ψ^0_μ evolves freely, the matrix elements in this sum are constant and can be evaluated at $t = 0$. Further, the operator $\exp[iap_x]$ is the translation operator through the vector $-\boldsymbol{a}$. Thus we can rewrite (24) as

$$(25)\qquad \langle \exp[ia\pi_x]\rangle_t = \sum_\nu \sum_\mu \exp[i(\alpha_\mu - \alpha_\nu)]\langle \psi^0_\nu(\boldsymbol{r}, 0), \psi^0_\mu(\boldsymbol{r}+\boldsymbol{a}, 0)\rangle .$$

If we first choose $\boldsymbol{a} = \boldsymbol{d}$ (the slit separation), then by (6)

$$\psi^0_2(\boldsymbol{r}+\boldsymbol{a}, 0) = \psi^0_2(\boldsymbol{r}+\boldsymbol{d}, 0) = \psi^0_1(\boldsymbol{r}, 0) .$$

Thus the matrix element with $\nu = 1$ and $\mu = 2$ is just $\langle \psi^0_1, \psi^0_1\rangle = \frac{1}{2}$. By the assumed nonoverlap of ψ^0_1 and ψ^0_2 (at $t = 0$), the other three matrix elements in (25) are zero and we find that

$$(26)\qquad \langle \exp[id\pi_x]\rangle_t = \tfrac{1}{2}\exp[-i\alpha] ,$$

where, as before, $\alpha = \alpha_1(\boldsymbol{r}) - \alpha_2(\boldsymbol{r})$. Since $\alpha = q\Phi$ (or $-e\Phi$ for electrons), this expectation value certainly varies with the applied magnetic field; that is, like $|\psi|^2$, its value when the field is present is different from that when the field is switched off, except when α is a multiple of 2π.

A similar analysis can be applied to the operator $\exp[-ia\pi_x]$ and hence to $\sin a\pi_x$ and $\cos a\pi_x$. All of these operators have expectation values that vary with the applied magnetic field.

There is in fact some neighborhood of $a = d$ for which the expectation values of these operators change with the AB field. The point is that, since the supports of ψ^0_1 and ψ^0_2 are assumed not to overlap at $t = 0$, there is a whole neighborhood of $a = d$ where three of the matrix elements in (25) are zero. The fourth matrix element is a continuous function of a and equals $[\exp[-i\alpha]]/2$ at $a = d$; therefore, it is close to $[\exp[-i\alpha]]/2$ for a close to $a = d$. Accordingly, the expectation value of $\exp[ia\pi_x]$ is shifted to a value close to $[\exp[-i\alpha]]/2$ for all a close to the slit separation d ([17]).

At first sight it is surprising that the expectation value $\langle \exp(ia\pi_x)\rangle$ can shift when none of the moments $\langle \pi^n_x\rangle$ do. Indeed, if we were to expand $\exp[ia\pi_x]$ in a power series

$$(27)\qquad \exp[ia\pi_x] = \sum (ia\pi_x)^n/n! ,$$

then a shift of $\langle \exp[ia\pi_x]\rangle$ would appear to require that at least one of the moments $\langle \pi^n_x\rangle$ also change. The resolution of this apparent paradox is that

([17]) In the limit of infinitely narrow slits there is a shift only for $a = d$; this is also the situation in ref. ([2]), which considers an infinite diffraction grating.

expansion (27) is valid only for a certain class of wave functions and our wave functions are not in this class. For example, in the case of the electric AB effect, $\pi = p$, and eq. (27) acting on a wave function $\psi(r)$ generates the Taylor series for $\psi(r + a)$, which converges only if $\psi(r + a)$ is analytic in a; but, since ψ_1^0 and ψ_2^0 are identically zero outside their supports, they certainly are not analytic.

AHARONOV, PENDLETON and PETERSON (²) actually considered the operator

$$D = (\sin a\pi_x)/a \tag{28}$$

since, in the limit $a \to 0$, this operator approaches π_x. In this sense D can be regarded as a generalization of π_x and a generalization whose value *is* affected by the AB fields. It is natural to ask whether its equation of motion offers any new insight into the AB effect. To conclude, we review briefly the discussion of this point by APP. For simplicity, we restrict attention to the electric case, although similar conclusions apply to the magnetic effect as well.

It is easily shown that in the Heisenberg picture operator (28) satisfies the equation of motion

$$\frac{dD}{dt} = -\frac{q}{2a}\{[\varphi(r + a, t) - \varphi(r, t)]\exp[iap_x] + [\varphi(r, t) - \varphi(r - a, t)]\exp[-iap_x]\}. \tag{29}$$

where $\varphi(r, t)$ is the electric potential and all operators are in the Heisenberg picture. In the limit that $a \to 0$, eq. (29) reduces to Ehrenfest's theorem:

$$\frac{dp_x}{dt} = -q\frac{\partial\varphi}{\partial x}.$$

However, with $a \neq 0$, (29) shows that the time evolution of the modular momentum (28) is determined by potential differences at points separated by the vector a. APP argue that this is what allows the modular momentum to shift, while all the moments of p_x (which depend on $\partial\varphi/\partial x$) are unchanged.

5. – Conclusion.

The two main results of this paper are that the introduction of magnetic or electric potentials in the AB experiment does not change the expectation value of any polynomial in the position r and kinetic momentum π, but that it does change the expectation value of the operators $\exp[ia\cdot\pi]$, $\sin a\cdot\pi$ and $\cos a\cdot\pi$ (for certain values of a). The first of these results appears superficially to contradict the observed shift of the AB interference pattern and, in the same way, the second result appears to contradict the first. On closer examination, both results, although surprising, are nevertheless perfectly consistent.

* * *

We would like to thank Profs. L. BAGGETT, R. BROOKS, K. JAGANNATHAN, and R. H. ROMER for helpful conversations, and Dr. M. PESHKIN for several useful comments on the manuscript. One of us (MDS) wishes to thank the Physics Department at Amherst College where much of this work was carried out.

● RIASSUNTO (*)

Si è ben stabilito che, come previsto da Aharonov e Bohm, le interferenze tra gli elettroni possono essere spostate dall'introduzione di potenziali elettromagnetici, anche se gli elettroni non entreranno mai nella regione nella quale i campi sono diversi da zero. In questo lavoro si prova che, anche se i comportamenti d'interferenza si spostano, nessuno degli impulsi della posizione degli elettroni $\boldsymbol{r}$, né del suo momento cinetico $\boldsymbol{\pi}$ è influenzato. D'altra parte, si prova che il valore atteso dell'operatore $\sin \boldsymbol{a}\cdot\boldsymbol{\pi}$ (con $\boldsymbol{a}$ un certo vettore fissato), che fu introdotto per primo da Aharonov, Pendleton e Peterson, si sposta.

(*) *Traduzione a cura della Redazione.*

Резюме не получено.