Screened Coulomb scattering in the eikonal approximation

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(Received 3 December 1976)

Scattering amplitudes and cross sections for screened Coulomb potentials are examined using the eikonal approximation. We use the method of h transforms to show that if the screening function is sufficiently smooth, then the screened cross section σ_ρ approaches the Coulomb cross section σ_C as the screening radius $\rho\to\infty$. On the other hand, for a sharply cut-off Coulomb potential σ_ρ does not approach σ_C as $\rho\to\infty$. These results agree with results obtained earlier using the Born approximation.

I. INTRODUCTION

In a previous paper[1] we established several results concerning scattering by screened potentials. In particular, we considered screened Coulomb potentials

$$V_\rho(r)=(\gamma/r)\alpha_\rho(r)\ , \tag{1.1}$$

where $\alpha_\rho(r)$ is a screening function characterized by a screening radius ρ and satisfying (at least)

$$\alpha_\rho(r)\to 0 \quad \text{as } r\to\infty,\ \ \rho \text{ fixed}\ ,$$

$$\alpha_\rho(r)\to 1 \quad \text{as } \rho\to\infty,\ \ r \text{ fixed}\ .$$

Using the Born approximation for the scattering amplitudes concerned, we established various conditions under which the scattering cross section for V_ρ approaches the Coulomb cross section as $\rho\to\infty$, and other conditions under which it does not.

Before summarizing the relevant results of Ref. 1 we must review our notation briefly. We denote the cross section for scattering an initial wave packet of shape ϕ into a small solid angle $d\Omega$ by $\sigma(d\Omega\leftarrow\phi)$. This is just $d\Omega$ times the usual differential cross section $d\sigma/d\Omega$,

$$\sigma(d\Omega\leftarrow\phi)=d\Omega\,\frac{d\sigma}{d\Omega}\ .$$

Standard manipulations (Ref. 2, pp. 49–51) let one express the cross section as

$$\sigma(d\Omega\leftarrow\phi)=d\Omega\int d^3p\,(p/p_z)|f(p\vec{u}\leftarrow\vec{p})\phi(\vec{p})|^2\ , \tag{1.2}$$

where $f(\vec{p}'\leftarrow\vec{p})$ is the scattering amplitude, $\vec{u}$ is a unit vector in the direction of $d\Omega$, and p_z is the component of $\vec{p}$ in the direction of the incident mean momentum $\vec{p}_0$.

For normal short-range potentials under ordinary conditions, the factors p/p_z and $f(p\vec{u}\leftarrow\vec{p})$ in (2.3) are essentially constant in the region where $\phi(\vec{p})$ is nonzero. Thus both factors can be taken outside the integral, with $\vec{p}$ replaced by $\vec{p}_0$. The remaining integral is the normalization integral for ϕ and the cross section reduces to the familiar $d\Omega|f(p_0\vec{u}\leftarrow\vec{p}_0)|^2$. However, as emphasized in Ref. 1, these familiar manipulations of (1.2) must be examined very carefully in the case of the screened Coulomb potential. This is because the screened Coulomb amplitude can contain oscillatory factors which oscillate more and more rapidly as the screening radius $\rho\to\infty$. Thus in our analysis of the screened Coulomb potential we must start from Eq. (1.2) for the cross section rather than the familiar $d\Omega|f|^2$.

Quantities which refer to the pure Coulomb potential we shall identify by a subscript capital C. Thus σ_C denotes the familiar Coulomb (or Rutherford) cross section and f_C the conventional Coulomb amplitude. The Born approximation to the Coulomb amplitude we denote by f_{CB}. As is well known, f_{CB} and f_C differ only by a phase factor and are given by

$$f_{CB}=-2\gamma/q^2\ ,$$

where q is the momentum transfer $q=2p\sin(\theta/2)$ and

$$f_C=f_{CB}\exp[2i\sigma_0-2i\gamma\ln(q/2)]\ ,$$

where $\sigma_0=\arg\Gamma(1+i\gamma)$ is the $l=0$ Coulomb phase shift.

Quantities which refer to the screened Coulomb potential (1.1), with screening radius ρ, we label with a subscript ρ. For convenience, we use units for which $\hbar$, m (the reduced mass), and p_0 (the incident mean momentum) are all equal to one.

In Ref. 1 we evaluated the screened Coulomb amplitude f_ρ *in Born approximation* and proved two principle results as follows:

(i) If the screening function $\alpha_\rho(r)$ is sufficiently smooth[3] then as $\rho\to\infty$ the screened cross section

σ_ρ approaches the Coulomb cross section σ_C

$$\lim_{\rho\to\infty} \sigma_\rho(d\Omega \to \phi) = \sigma_C(d\Omega \to \phi) \ , \qquad (1.3)$$

(ii) but if $\alpha_\rho(r)$ is the sharp cutoff

$$\alpha_\rho(r) = \theta(\rho - r) \ ,$$

then σ_ρ does *not* approach σ_C and in fact

$$\lim_{\rho\to\infty} \sigma_\rho(d\Omega \to \phi) = \tfrac{3}{2}\sigma_C(d\Omega \to \phi) \ . \qquad (1.4)$$

These two results were proved by computing the appropriate screened amplitude f_ρ in Born approximation and inserting it in Eq. (1.2) for the cross section. In the smooth case we found that, as $\rho \to \infty$, f_ρ approaches f_C uniformly for all $\vec{p}$ in the region of integration; which immediately implies the limit (1.3). However, for the sharp cutoff we found that f_ρ approaches f_C *plus an oscillating term*. We showed that this extra term corresponds to scattering from the discontinuity in the potential at $r=\rho$, and that, no matter how large we make ρ, those packets whose impact parameter b is of order ρ are scattered by the discontinuity and contribute exactly the extra $\frac{1}{2}\sigma_C$ to the cross section. Although both of our results were proved only in Born approximation, we conjectured that it is reasonable to expect them to be true for the exact cross sections as well.

In his lectures at the Boulder Summer Institute in 1958,[4] Glauber stated a result which throws considerable doubt on our conjecture that result (1.4) is exactly true. He computed the amplitude f_ρ for the sharp cutoff in *eikonal approximation* and stated that the cross section σ_ρ *does* converge to σ_C as $\rho \to \infty$.[5] Now, the eikonal approximation is generally believed to be especially reliable for large impact parameters, which is precisely where the extra $\frac{1}{2}\sigma_C$ in our result (1.4) comes from. Thus if it is correct that the extra $\frac{1}{2}\sigma_C$ is absent from the eikonal approximation, it seems likely that its presence in our previous analysis reflects our use of an unsuitable approximation (the Born approximation) rather than the true state of affairs.

It is clearly desirable to examine closely the behavior of screened Coulomb potentials using the eikonal approximation; and this is what we do in this paper. We make an asymptotic expansion of the amplitudes using the method of "h transforms" as described in Chap. 4 of the recent and beautiful book of Bleistein and Handelsman.[6] We show first that for smooth screening functions the cross section σ_ρ *does* converge to the Coulomb cross section σ_C; that is, our result (1.3) holds in eikonal approximation. Second, we show that for the sharp cutoff, σ_ρ does *not* converge to σ_C, but converges instead to $\frac{3}{2}\sigma_C$ as in our Eq. (1.4); that is, our result (1.4) is also correct in eikonal approximation, and Glauber's claim is incorrect. That both of our results (1.3) and (1.4) are also correct in eikonal approximation gives strong support, we feel, to our conjecture that they are in fact exactly true.

The main body of this paper can be briefly described as follows: In Sec. II we briefly review the eikonal approximation and the method of h transforms. In Sec. III we prove the result (1.3) (in eikonal approximation) for a smooth screening function of the form $\alpha(r/\rho)$, where $\alpha(\xi)$ has four continuous derivatives that decrease like the following powers as $\xi \to \infty$:

$$\alpha^{(n)}(\xi) = O(\xi^{-2-n-\epsilon}) \text{ as } \xi \to \infty \ ,$$

for some $\epsilon > 0$ and $n = 0, 1, 2, 3, 4$. These conditions are more restrictive than (and include) the conditions used in Ref. 1; they are probably more restrictive than is necessary. However, as mentioned in Ref. 1, a precise set of necessary conditions is probably not very interesting to know.

In Sec. IV we give the corresponding analysis for the sharp cutoff and prove the result (1.4). We show that in this case the integral defining the amplitude has one more critical point than in the smooth case, and that it is this extra critical point which contributes the extra $\frac{1}{2}\sigma_C$ in (1.4). In Sec. V we sketch the proof of some of the estimates used in Secs. III and IV.

To conclude this introduction we should emphasize that we do not claim that our conclusion—that smooth screening functions yield the Coulomb cross section when $\rho \to \infty$, while the sharp cutoff does not—is particularly surprising. Nonetheless it is important to establish precisely what screening functions can be safely used, and we offer the present work as a step in this direction. In addition, we suspect that the powerful method of h transforms is not well known to many physicists, and it is our hope that the present work will help to make it more widely known.

II. EIKONAL APPROXIMATION AND h TRANSFORMS

A. Eikonal approximation

The eikonal approximation to the amplitude for a spherical potential $V_\rho(r)$ is (see Newton,[7] Eq. 18.32)

$$f_\rho = -i \int_0^\infty db\, bJ_0(bq)\{\exp[2i\gamma\,\chi_\rho(b)] - 1\} \ , \qquad (2.1)$$

where the eikonal "phase shift" is given by

$$\gamma\chi_\rho(b) = -\int_b^\infty dr\, r V_\rho(r)(r^2 - b^2)^{-1/2} \ .$$

Since the phase shift is proportional to the potential we have divided out a factor of γ explicitly.

To exhibit more clearly the behavior of f_ρ as $\rho \to \infty$ we rewrite (2.1), making the change of variables $b = \rho t$, as

$$f_\rho = -i \frac{\lambda^2}{q^2} \int_0^\infty dt\, t J_0(\lambda t)[e^{2i\gamma\Delta(t)} - 1] \; . \tag{2.2}$$

Here we have introduced the variable

$$\lambda = q\rho \; , \tag{2.3}$$

and have rewritten the phase shift χ_ρ as

$$\begin{aligned} \Delta(t) &\equiv \chi_\rho(\rho t) \\ &= -\int_{\rho t}^\infty dr\, \alpha(r/\rho)(r^2 - \rho^2 t^2)^{-1/2} \\ &= -\int_t^\infty d\xi\, \alpha(\xi)(\xi^2 - t^2)^{-1/2} \; , \end{aligned} \tag{2.4}$$

which is independent of ρ, as our notation implies.

Finally, it proves convenient to rewrite (2.2) using a single integration by parts. Using the identity

$$x J_0(x) = \frac{d}{dx}[x J_1(x)]$$

in (2.2), we obtain as our final form for the eikonal approximation to the screened amplitude

$$f_\rho = f_{\rm CB} \lambda H(\lambda) \; , \tag{2.5}$$

where

$$H(\lambda) = \int_0^\infty dt J_1(\lambda t) t \Delta'(t) e^{2i\gamma\Delta(t)} \; . \tag{2.6}$$

Here we have assumed (what we shall check explicitly later) that the integrals converge and that the end-point terms of the integration by parts are zero.

Since we shall always be concerned with values of q which lie in a compact interval excluding $q = 0$, the limit of interest ($\rho \to \infty$) is equivalent to the limit $\lambda = \rho q \to \infty$. Therefore, we must find the asymptotic form of the integral (2.6) as $\lambda \to \infty$.

B. h transforms

The integral (2.6) is in exactly the form required to apply the method of h transforms. This method, which is described in Chap. 4 of Bleistein and Handelsman,[6] establishes the asymptotic form as $\lambda \to +\infty$ (or zero) of an integral of the form

$$H(\lambda) = \int_0^\infty dt\, h(\lambda t) f(t) \; . \tag{2.7}$$

If the two functions h and f have Mellin transforms that are sufficiently well behaved then this integral can be rewritten using the Parseval relation for Mellin transforms as

$$H(\lambda) = (2\pi i)^{-1} \int_{r - i\infty}^{r + i\infty} dz\, \lambda^{-z} M[h, z] M[f, 1 - z] \; . \tag{2.8}$$

Here $M[g, z]$ denotes the Mellin transform of a function g,

$$M[g, z] = \int_0^\infty dt\, g(t) t^{z-1} \; ,$$

and the contour of integration in (2.8) is the vertical line $\text{Re} z = r$ in the complex plane of z. It will be seen that the λ dependence of $H(\lambda)$ has been isolated in the term λ^{-z} in the integrand of (2.8).

Under suitable conditions (of convergence, etc.) we can say that the larger the value of r in (2.8) the more rapidly $H(\lambda)$ goes to zero as $\lambda \to \infty$. Further, if the integrand in (2.8) is meromorphic in z, then we may be able to move the contour of integration $\text{Re} z = r$ in (2.8) to the right to some position $\text{Re} z = s$. Each time the contour crosses a pole at z_i we would pick up a term of the form λ^{-z_i} times the residue of the two Mellin transforms. This would produce a finite asymptotic expansion of the form

$$H(\lambda) = -\sum_i \beta_i \lambda^{-z_i} + (2\pi i)^{-1} \int_{s - i\infty}^{s + i\infty} dz\, \lambda^{-z} M[h, z] M[f, 1 - z] \; , \tag{2.9}$$

where β_i is the residue of the two Mellin transforms at the pole z_i, and the sum runs over all poles z_i between the lines $\text{Re} z = r$ and $\text{Re} z = s$. (We assume, as will prove to be the case, that all poles are simple.)

To justify the asymptotic expansion (2.9) we shall have to examine the behavior of the two Mellin transforms as $\text{Im} z \to \pm\infty$ and establish their analytic properties. These considerations are discussed in some generality by Bleistein and Handelsman (Ref. 6, pp. 106–117). We shall examine them only as they are needed in our calculations and shall lean on the discussion of Bleistein and Handelsman.

III. SMOOTH SCREENING

We first use the formalism of Sec. II to estimate the amplitude f_ρ for a smoothly screened potential $V_\rho = (\gamma/r)\alpha(r/\rho)$, where we require that the screening function $\alpha(\xi)$ have four continuous derivatives,

$$\alpha(\xi) \in C^4[0, \infty) \; , \tag{3.1}$$

that $\alpha(0) = 1$, and that

$$\alpha^{(n)}(\xi) = O(\xi^{-2-n-\epsilon}) \quad \text{as } \xi \to \infty \; , \tag{3.2}$$

for $n = 0, 1, 2, 3, 4$ and some $\epsilon > 0$. As mentioned in the introduction, these conditions could probably be relaxed, but the precise set of necessary conditions for our result is probably not very interesting, since we are only working in an approximation anyway.

We shall prove that as $\rho \to \infty$ the screened amplitude has the form

$$f_\rho = f_C e^{2i\zeta(\rho)} + O(\rho^{-\beta}) \ . \tag{3.3}$$

Here f_C denotes the exact Coulomb amplitude (including its correct phase), $\zeta(\rho)$ is the expected ρ-dependent phase factor [Eq. (3.19) below and Ref. 8, Eq. (4.2)] and β is any number less than one. The form (3.3) is uniform for all energies and momentum transfers in any closed finite intervals excluding $E=0$ and $q=0$. As discussed in the Introduction (and Ref. 1) this guarantees the desired limit for the screened cross section

$$\sigma_\rho(d\Omega) \to \sigma_C(d\Omega),$$

for any cone $d\Omega$ excluding the forward direction.

It should be emphasized that, unlike the Born result of Ref. 1, the eikonal result (3.3) includes the exact Coulomb amplitude with all the correct phase factors. This illustrates the marked superiority of the eikonal approximation in the present context. Nevertheless, as far as cross sections are concerned, the two approximations give the same results for the limits as $\rho \to \infty$.

Let us now prove the result (3.3). As seen in Sec. II the screened amplitude has the form (2.5), $f_\rho = f_{CB}\lambda H(\lambda)$, where f_{CB} is the Coulomb amplitude in Born approximation and $\lambda = \rho q$. The function $H(\lambda)$ is the Hankel transform

$$H(\lambda) = \int_0^\infty dt J_1(\lambda t) f(t) \ , \tag{3.4}$$

where

$$f(t) = t\Delta'(t) e^{2i\gamma\Delta(t)} \ , \tag{3.5}$$

and the phase shift $\Delta(t)$ is given in terms of the screening function $\alpha(\xi)$ by the integral (2.4).

To apply the method of h transforms to (3.4) we must study the Mellin transforms $M[J_1, z]$ and $M[f, 1-z]$ of J_1 and f. The transform of J_1 is known explicitly (Ref. 6, p. 414) to be

$$M[J_1, z] = 2^{z-1}\Gamma(\tfrac{1}{2}(z+1))/\Gamma(\tfrac{1}{2}(3-z)) \ . \tag{3.6}$$

This is meromorphic for all z with poles at the negative odd integers and has the asymptotic form (Ref. 6, Eq. 3.2.41)

$$M[J_1, z] = O(y^{x-1}) \ \text{ as } |y| \to \infty \ , \tag{3.7}$$

where $z = x + iy$.

The Mellin transform of f is defined by the integral

$$M[f, 1-z] = \int_0^\infty dt\, t^{-z} f(t) \ . \tag{3.8}$$

The convergence of this integral depends on the smoothness of $f(t)$ and its behavior at the end points $t=0$ and ∞. These in turn depend on the properties of the phase shift $\Delta(t)$ and thence on the screening function $\alpha(\xi)$. The calculation of these properties is a straight-forward but tedious exercise, which we sketch in Sec. V below.[9] The first results are that, subject to conditions (3.1) and (3.2) on α, the phase shift Δ has a continuous derivative and that

$$f(t) = \begin{cases} t^{2i\gamma} e^{2i\gamma K}[1 + O(t \ln t)] & \text{as } t \to 0 \ , \\ O(t^{-2-\epsilon}) & \text{as } t \to \infty \ , \end{cases} \tag{3.9}$$

where

$$K = \int_0^\infty d\xi\, \alpha'(\xi) \ln 2\xi \ . \tag{3.10}$$

These results justify the integration by parts which led to the form (2.5) for the amplitude, and show that the integral (3.8) for the Mellin transform $M[f, 1-z]$ is convergent and analytic in the strip

$$\{-1 < \mathrm{Re} z < 1\} \ .$$

We now need to continue $M[f, 1-z]$ into the region $\{\mathrm{Re} z > 1\}$ and to find estimates on its behavior as $\mathrm{Im} z = y \to \infty$. To this end we integrate (3.8) by parts. In view of the behavior (3.9) it is natural to rewrite (3.8) as

$$M[f, 1-z] = \int_0^\infty dt\, t^{-z+2i\gamma} g(t) \ , \tag{3.11}$$

where $g(t) = t^{-2i\gamma} f(t)$, (that is, we factor out of f its dominant behavior as $t \to 0$). The calculations in Sec. V show that, again subject to the smoothness conditions (3.1) and (3.2), $g(t)$ can be differentiated twice and satisfies

$$g'(t) = \begin{cases} O(\ln t) & \text{as } t \to 0 \ , \\ O(t^{-3-\epsilon}) & \text{as } t \to \infty \ , \end{cases} \tag{3.12}$$

$$g''(t) = \begin{cases} O(t^{-1} \ln t) & \text{as } t \to 0 \ , \\ O(t^{-4-\epsilon}) & \text{as } t \to \infty \ . \end{cases} \tag{3.13}$$

Armed with these estimates we see easily that (3.10) can be integrated by parts twice to give

$$M[f, 1-z] = \frac{1}{(z-1-2i\gamma)(z-2-2i\gamma)} \times \int_0^\infty dt\, t^{2-z+2i\gamma} g''(t) \ . \tag{3.14}$$

The important features of (3.14) are as follows: First, the integral is convergent and analytic in the larger strip

$$\{-1 < \mathrm{Re} z < 2\} \ . \tag{3.15}$$

Thus (3.14) defines the continuation of $M[f, 1-z]$ as a meromorphic function into this wider strip. This continuation is sufficient for our purposes. Second, the continuation (3.14) of $M[f, 1-z]$ has a simple pole at $z = 1 + 2i\gamma$ whose residue is easily evaluated as $-g(0) = -\exp(2i\gamma K)$. Finally it is clear from (3.14) that as $\mathrm{Im} z = y \to \infty$,

$$M[f, 1-z] = O(y^{-2}) \text{ as } |y| \to \infty \quad (3.16)$$

anywhere in the strip (3.15).

With these properties of the Mellin transforms of J_1 and f we are ready to estimate the integral $H(\lambda) = \int_0^\infty dt J_1(\lambda t) f(t)$. First it is easy to check that the transforms are well enough behaved to allow us to rewrite $H(\lambda)$ using the Parseval relation (2.8)

$$H(\lambda) = (2\pi i)^{-1} \int_{r-i\infty}^{r+i\infty} dz\, \lambda^{-z} M[J_1, z] M[f, 1-z] \ , \quad (3.17)$$

provided the contour $\text{Re}z = r$ lies in the strip $\{-1 < \text{Re}z < 1\}$. [For conditions which justify the Parseval relation (3.17) see Ref. 6, p. 108.]

The asymptotic forms (3.7) and (3.16) show that the contour in (3.17) can be moved to the right as close as we please to the line $\text{Re}z = 2$. In doing this we must pass the simple pole of $M[f, 1-z]$ at $z = 1 + 2i\gamma$. We therefore obtain the finite asymptotic expansion [cf. (2.9)]

$$H(\lambda) = 2^{2i\gamma} \frac{\Gamma(1+i\gamma)}{\Gamma(1-i\gamma)} e^{2i\gamma K} \lambda^{-(1+2i\gamma)} + O(\lambda^{-s}) \quad (3.18)$$

for any $s < 2$. The ratio of Γ functions here will be recognized as $\exp(2i\sigma_0)$, where σ_0 is the $l=0$ Coulomb phase shift. The number K is given by (3.10) and is easily rewritten as

$$\gamma K = \gamma \ln\rho + \zeta(\rho) + O(\rho^{-1}\ln\rho) \ ,$$

where $\zeta(\rho)$ is the ρ-dependent phase factor introduced in Ref. 8, Eq. (4.2),

$$\zeta(\rho) = -\gamma \int_{1/2}^{\infty} dr\, \alpha(r/\rho)/r \ . \quad (3.19)$$

If we now replace λ by ρq in (3.18) and substitute into the amplitude $f_\rho = f_{\text{CB}} \lambda H(\lambda)$ we find

$$f_\rho = f_{\text{CB}} e^{2i\sigma_0} e^{-2i\gamma \ln(q/2)} e^{2i\zeta(\rho)} + O(\rho^{-\beta})$$
$$= f_C e^{2i\zeta(\rho)} + O(\rho^{-\beta})$$

for any $\beta < 1$. This is exactly the required result (3.3).

IV. SHARP CUTOFF

If the screening function $\alpha(\xi)$ is the sharp cutoff

$$\alpha(\xi) = \theta(1-\xi) \ ,$$

then the corresponding eikonal phase shift (2.4) can be explicitly evaluated as

$$\Delta(t) = \begin{cases} \ln t - \ln[1 + (1-t^2)^{1/2}] \ , & t \leq 1 \ , \\ 0 \ , & t > 1 \ . \end{cases} \quad (4.1)$$

We can again put the amplitude $f_\rho = f_{\text{CB}} \lambda H(\lambda)$ where $H(\lambda)$ is now the Hankel transform

$$H(\lambda) = \int_0^1 dt\, J_1(\lambda t) f(t) \quad (4.2)$$

and $f(t)$ can be calculated explicitly as

$$f(t) = (1-t^2)^{-1/2} t^{2i\gamma} [1 + (1-t^2)^{+1/2}]^{-2i\gamma} \ . \quad (4.3)$$

There are two important differences between (4.2) and the corresponding integral (3.4) for the smooth case. First the upper limit of integration in (4.2) is $t=1$ not $t=\infty$ as in (3.4); second, the function $f(t)$ has a singularity at $t=1$. These mean that $t=1$ may be (and, in fact, is) a critical point (Ref. 6, p. 84). As we shall discuss further, it was the neglect of this extra critical point that lead Glauber to his incorrect conclusion that $\sigma_\rho \to \sigma_C$ as $\rho \to \infty$ for the sharp cutoff.

As $\lambda \to \infty$, we must anticipate that $H(\lambda)$ in (4.2) will have contributions from both of the critical points $t=0$ and 1. It is convenient to isolate these contributions using the method of neutralizers. We let $\nu(t)$ be any C^∞ function such that

$$\nu(t) = \begin{cases} 1 \text{ in a neighborhood of } t=0 \\ 0 \text{ in a neighborhood of } t=1, \end{cases}$$

and write

$$f(t) = f(t)\nu(t) + f(t)[1-\nu(t)]$$
$$= f_0(t) + f_1(t) \ .$$

We can then make corresponding decompositions of $H(\lambda)$ and of f_ρ; for example,

$$H(\lambda) = H_0(\lambda) + H_1(\lambda) \ ,$$

where $H_i(\lambda) = \int_0^1 J_1(\lambda t) f_i(t) (i = 0 \text{ or } 1)$ and the two terms H_0 and H_1 can be analyzed separately.

The contribution to the amplitude from the critical point at $t=0$ is determined by the function $f_0(t)$. From the explicit form (4.3) it is easy to see that $f_0(t)$ has all the properties of the function $f(t)$ for the smooth cutoff discussed in Sec. II. Accordingly the corresponding contribution f_ρ^0 to the amplitude has exactly the form found in Sec. III:

$$f_\rho^0 = f_C e^{2i\zeta(\rho)} + O(\rho^{-\beta}) \quad (4.4)$$

for any $\beta < 1$.

The contribution from the critical point at $t=1$ is given by the function

$$H_1(\lambda) = \int_0^1 dt\, J_1(\lambda t) f_1(t) \ .$$

Because $f_1(t) = 0$ in a neighborhood of $t=0$, we can use the large argument expansion of J_1,

$$J_1(\lambda t) = (2/\pi\lambda t)^{1/2} \cos(\lambda t - 3\pi/4) + O(\lambda^{-3/2})$$

which is uniform in the region where $f_1(t) \neq 0$. This gives

$$H_1(\lambda) = (2/\pi\lambda)^{1/2}\int_0^1 dt\cos(\lambda t - 3\pi/4)t^{-1/2}f_1(t) + O(\lambda^{-3/2}). \quad (4.5)$$

The singularities of the function $t^{-1/2}f_1(t)$ at $t=1$ originate in the factors $(1-t)^{1/2}$ in (4.3). It is convenient to move these singularities to the origin by substituting $1-t=u$. We can then isolate the dominant singularity [the first factor $(1-t)^{-1/2}$] by writing

$$t^{-1/2}f_1(t) = u^{-1/2}g(u) .$$

The function $g(u)$ can be written down explicitly but all we shall need to know is that $g(0)=2^{-1/2}$, that $g(u)\equiv 0$ in a neighborhood of $u=1$, and that $g(u)$ is infinitely differentiable on $(0,1]$ with

$$g'(u) = O(u^{-1/2}) \text{ as } u\to 0 . \quad (4.6)$$

[The singularity at $u=0$ comes from the factor $(1-t)^{1/2}$ in (4.3).]

If we now split the cosine in (4.5) as the sum of two exponentials, we obtain

$$H_1(\lambda) = (2/\pi\lambda)^{1/2}\exp i(\tfrac{3}{4}\pi-\lambda)\int_0^1 du\, u^{-1/2}e^{i\lambda u}g(u) + (\text{c.c. and } \gamma\to-\gamma) + O(\lambda^{-3/2}) , \quad (4.7)$$

where the second term is obtained from the first by complex conjugation and the replacement of γ by $-\gamma$. The integral (4.7) can be estimated by a method of Erdelyi.[10] Since $g(u)$ is differentiable, we can integrate by parts using as integral of $u^{-(1/2)}e^{i\lambda u}$ the function

$$h(u) = \int_{i\infty}^{u} du'\, u'^{-1/2}e^{i\lambda u'} . \quad (4.8)$$

This function is easily seen to satisfy

$$h(0) = -e^{i\pi/4}(\pi/\lambda)^{1/2} ,$$

and we show in Sec. V that

$$|h(u)| < K\lambda^{\eta-1}u^{\eta-1/2} \quad (4.9)$$

for any $\eta>0$.

If we now perform the integration in (4.7) by parts, we find

$$\int_0^1 du\, u^{-1/2}e^{i\lambda u}g(u) = e^{i\pi/4}(\pi/2\lambda)^{1/2} - \int_0^1 du\, h(u)g'(u) . \quad (4.10)$$

Using the bounds (4.6) and (4.9) it is easily seen that the remaining integral is $O(\lambda^{-1+\eta})$ for any $\eta>0$. Inserting (4.10) into (4.7) we obtain

$$H_1(\lambda) = -\lambda^{-1}\cos\lambda + O(\lambda^{\eta-3/2}) .$$

Finally, the corresponding contribution to the amplitude is

$$f_\rho^1 = f_{CB}\lambda H_1(\lambda) = -f_{CB}\cos\rho q + O(\rho^{\eta-1/2}) . \quad (4.11)$$

We have now estimated both contributions, (4.4) and (4.11), to the amplitude for the sharp cutoff. We can combine them to give the final answer

$$f_\rho = f_\rho^0 + f_\rho^1 = f_C e^{2i\zeta(\rho)} - f_{CB}\cos\rho q + O(\rho^{\eta-1/2}) \quad (4.12)$$

for any $\eta>0$. This answer is very similar to the Born result of Ref. 1, and all of the discussion of Ref. 1 applies to it. The first term is the desired Coulomb amplitude (including all the correct phases, as in the smooth case). The second term has precisely the form found for the Born approximation. If we evaluate the scattered wave for a *fixed* incident packet $\phi(\vec{p})$ then the contribution from this second term goes to zero as $\rho\to\infty$.[11] However, the measurement of the cross section σ_ρ involves many different packets $\phi_b(\vec{p})$ with random impact parameters $\vec{b}$. We emphasized in Ref. 1 that, however large we make the cutoff radius ρ, certain packets pick up an appreciable scattered contribution from the second term in (4.12). For a given direction of observation θ, the packets for which this contribution is appreciable have

$$b \approx \rho\cos(\theta/2) . \quad (4.13)$$

This is just the condition that the packets undergo specular reflection off the discontinuity in the potential at $r=\rho$, as discussed in Ref. 1 (Fig. 1 and following Eq. 2.14).

However large we make the cutoff radius ρ there are two contributions to the observed cross section σ_ρ. Those packets with small b contribute through the first term in (4.12); those with b satisfying the specular condition (4.13) contribute through the second term. These two contributions to σ_ρ can be evaluated as in Ref. 1. The first is exactly σ_C, while the second gives the additional $\frac{1}{2}\sigma_C$. Thus $\sigma_\rho\to\frac{3}{2}\sigma_C$ exactly as in the Born result of Ref. 1. We see that, as anticipated, it was the neglect of the second term in (4.12) which led Glauber to his incorrect conclusion that $f_\rho\to f_C\exp(2i\zeta)$ and hence that $\sigma_\rho\to\sigma_C$ for the sharp cutoff.

V. SOME PROOFS

A. Eikonal phase for smooth screening

We need the behavior as $t\to 0$ and ∞ of the phase shift $\Delta(t)$ for a smooth screening function $\alpha(r/\rho)$ satisfying (3.1) and (3.2). The integral (2.4) which defines $\Delta(t)$ can be rewritten as

$$\Delta(t) = -\int_1^\infty d\eta\, \alpha(\eta t)(\eta^2-1)^{-1/2} .$$

Since α is four times differentiable and satisfies the bounds (3.2) it is immediately clear that $\Delta(t)$ is likewise four times differentiable with derivatives given by

$$\Delta^{(n)}(t) = -\int_1^\infty d\eta\, \eta^n \alpha^{(n)}(\eta t)(\eta^2-1)^{-1/2} . \qquad (5.1)$$

Inserting the bounds (3.2) for $\alpha^{(n)}$ we obtain the bounds

$$\Delta^{(n)}(t) = O(t^{-2-n-\epsilon}) \text{ as } t\to\infty \qquad (5.2)$$

for $n=0,\ldots,4$.

To find the behavior of $\Delta^{(n)}(t)$ as $t\to 0$ we rewrite (5.1) as

$$\Delta^{(n)}(t) = -t^{-n}\int_t^\infty d\xi\, \xi^n \alpha^{(n)}(\xi)(\xi^2-t^2)^{-1/2} . \qquad (5.3)$$

We next remark that the following lemma is easily proved.

Lemma. Let

$$G(t) = \int_t^\infty d\xi\, g(\xi)(\xi^2-t^2)^{-1/2} ,$$

where $g(\xi)$ is C^1 on $[0,\infty)$ and both $g(\xi)$ and $g'(\xi)$ are $O(\xi^{-1-\epsilon})$as $\xi\to\infty$. Then

$$G(t) = -g(0)\ln t - \int_0^\infty d\xi\, g'(\xi)\ln 2\xi + O(t\ln t)$$

as $t\to 0$.

Repeated application of this lemma to $\Delta(t)$ and its first three derivatives gives the following estimates as $t\to 0$:

$$\Delta(t) = \ln t + K + O(t\ln t) , \qquad (5.4)$$

where K is given by (3.10)

$$\Delta'(t) = t^{-1}[1+O(t\ln t)] , \qquad (5.5)$$

$$\Delta''(t) = -t^{-2}[1+O(t\ln t)] , \qquad (5.6)$$

$$\Delta'''(t) = 2t^{-3}[1+O(t\ln t)] . \qquad (5.7)$$

The function $f(t)$ of Sec. III was defined in terms of $\Delta(t)$ by (3.5), while $g(t)$ was defined as $t^{-2i\gamma}f(t)$. Insertion of the bounds (5.2) and (5.4)–(5.7) into these definitions leads, after some straightforward algebra, to the bounds (3.9), (3.12), and (3.13) used in Sec. III.

B. Bound used for sharp cutoff

The function $h(u)$ as defined in (4.8) can be rewritten as

$$h(u) = -ie^{i\lambda u}\int_0^\infty dv\, e^{-\lambda v}(u+iv)^{-1/2} .$$

Therefore

$$|h(u)| \leq \int_0^\infty dv e^{-\lambda v}(u^2+v^2)^{-1/4} . \qquad (5.8)$$

For any two positive numbers x and y, we note that $x+y \geq x^{1-\epsilon}y^\epsilon$ for any ϵ with $0\leq\epsilon\leq 1$. Inserting this inequality (with $x=u^2$, $y=v^2$, and $\epsilon=2\eta$) into (5.8) we find

$$|h(u)| \leq \lambda^{\eta-1}u^{\eta-1/2}\int_0^\infty dw\, e^{-w}w^{-\eta}$$

which is the bound (4.9).

ACKNOWLEDGMENTS

We are grateful to David Goodmanson for several useful discussions and especially grateful to Professor Norman Bleistein of Denver University for introducing us to the method of h transforms and for taking the time to sharpen our understanding of asymptotic techniques in general.

*Supported in part by a Faculty Research Grant from Bates College.

[1]M. D. Semon and J. R. Taylor, J. Math. Phys. 17, 1366 (1976).

[2]J. R. Taylor, *Scattering Theory* (Wiley, New York, 1972).

[3]We shall discuss the precise conditions on $\alpha_p(r)$ below.

[4]R. J. Glauber, *High Energy Collision Theory* in *Lectures in Theoretical Physics*, edited by W. E. Brittin and L. G. Dunham (Interscience, New York, 1959), Vol. 1, p. 315.

[5]We should emphasize that the discussion of the cutoff Coulomb potential is just one small example in these celebrated lectures of Glauber. Nevertheless, in this example it is clearly implied that σ_ρ does approach σ_C for the sharply cutoff Coulomb potential. In connection with Eq. (122) it is stated that f_ρ approaches f_C times a phase factor, and the conclusion is drawn that, for angles $\theta \gg 1/p\rho$, "the scattered intensity $|f(\theta)|^2$ indeed follows the Rutherford formula." In fact, as we shall see below, neither of these conclusions is correct for the sharp cutoff in eikonal approximation.

[6]N. Bleistein and R. A. Handelsman, *Asympototic Expansions of Integrals* (Holt, Rinehart, and Winston, New York, 1975).

[7]R. G. Newton, *Scattering Theory of Waves and Particles* (McGraw Hill, New York, 1966).

[8]J. R. Taylor, Nuovo Cimento B 23, 313 (1974).

[9]We found it helpful to begin by considering the particular case of exponential screening, $\alpha(\xi)=\exp(-\xi)$, for which the phase shift can be explicitly evaluated as $\Delta(t) = -K_0(t)$. [See Gradshteyn and Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1965), p. 959, Eq. 8.] In this case one can avoid the tedious estimates of Sec. V, and the h-transform method can be applied using standard properties of the Bessel function $K_0(t)$. [See, for example, M. Abramowitz and I. Stegun, *Handbook of Mathema-*

tical Functions (Nat. Bur. Stand., Appl. Math. Series 55, 1964), pp. 375–378].

[10]A. Erdelyi, *Asympototic Expansions* (Dover, New York, 1956), Sec. 3.4; or see Ref. 6, p. 89.

[11]In this sense, Glauber's claim that $f_\rho \to f_C \exp(2i\zeta)$ is correct. The result illustrates (in eikonal approximation) the exact result of Ref. 8 that f_ρ does approach $f_C \exp(2i\zeta)$ when considered as a distribution.