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SYMMETRY AND SOLUTIONS TO THE HELMHOLTZ EQUATION INSIDE AN EQUILATERAL TRIANGLE

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Abstract. Solutions to the Helmholtz equation within an equilateral triangle which solve either the Dirichlet or Neumann problem are investigated. This is done by introducing a pair of differential operators, derived from symmetry considerations, which demonstrate interesting relationships among these solutions. One of these operators preserves the boundary condition while generating an orthogonal solution and the other leads to a bijection between solutions of the Dirichlet and Neumann problems. *MSC*: 35J05, 35B06
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1. Introduction

The solutions to many important physical problems, such as electromagnetic waves in waveguides [9], lasing modes in nanostructures [3], the electronic structure of graphene [8] and the quantum eigenvalues and eigenfunctions for various potential energies [5] are obtained by solving the ubiquitous Helmholtz equation:

$$\nabla^2\psi + k^2\psi = 0. \tag{1}$$

Studying the solutions to this equation is both a very old problem and one which continues to be an area of active research ([2], [14], [16]). In this paper, we discuss the solutions to this equation when the region of interest is an equilateral triangle and we consider two different boundary conditions: Dirichlet and Neumann. Although the explicit solutions in these cases are well-known, ([3], [5], [6], [11], [12], [13]) we present an alternative and more elegant framework for understanding them. These insights follow from properties of two differential operators which exploit the symmetry of the boundary.

The first operator we introduce, Θ_s, will preserve the boundary condition but transforms the solution to an orthogonal one. For degenerate solutions this becomes a (graded) involution on the solution space, while non-degenerate solutions like the ground state must transform to the trivial solution. Treating this as an additional constraint, we are able to find a novel derivation of the ground state solution and its harmonics.

In addition to insights for fixed boundary condition, one of the operators we introduce transforms the boundary condition, giving new insight between the Dirichlet and Neumann Problems. In particular, this provides a novel and constructive demonstration of the fact that they have the same spectrum (see [13], [15]).

In Section 2 the Dirichlet and Neumann Problems are introduced and an important known results is discussed. The aforementioned differential operators are introduced in Section 3, and their properties are explored. In Section 5 these results are reviewed with some concluding remarks.

2. Background and Elementary Results

Explicit solutions using elementary functions only exist for four two-dimensional domains: the rectangle and the three special triangles $((\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}), (\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2})$, and $(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}))$ [9]. Thus, for regular polygons, only the triangle and square admit explicit solutions. For other regular polygons it is known that the ground state (lowest eigenvalue) solution is not analytic in a neighborhood of the vertices, and only estimates are known for the energy of the ground state solution [1], [10].

While there are many beautiful results that hold for more general domains, we will make the most use of a theorem attributed to Lamé, using the statement (and referring the reader to the proof) given in reference [12]:

Theorem 1 *(Lamé) Suppose that $T(x, y)$ is a solution to the Helmholtz equation which can be represented by the trigonometric series*

$$T(x,y) = \sum_i \big(A_i \sin(\lambda_i x + \mu_i y + \alpha_i) + B_i \cos(\lambda_i x + \mu_i y + \beta_i)\big), \qquad (2)$$

with $\lambda_i^2 + \mu_i^2 = k^2$. Then

1. *$T(x, y)$ is antisymmetric about any line along which it vanishes;*
2. *$T(x, y)$ is symmetric about any line along which its normal derivative, $\frac{\partial T}{\partial \nu}$, vanishes.*

The reason this result is so valuable to the current work is that it interprets a nodal line as a line of anti-symmetry and an anti-nodal line as a line of symmetry. This correspondence with symmetry will become central in our treatment.

In particular, this is why we will focus on the Dirichlet and Neumann Problem. For the Dirichlet Problem, a solution must vanish on the boundary. In light of Theorem 1, this means that the solutions are antisymmetric across the boundary. For the Neumann Problem, the solutions must have the normal derivative vanish along the boundary. Once again, by Theorem 1, this means that the solutions are symmetric across the boundary.

In addition to understanding the boundary conditions as either symmetric or antisymmetric, we can also decompose any solution into an even and odd part. For concreteness, we will draw our triangle with vertices at the roots of unity. With this choice made, we will now be able to conclude that a function which is even in y is orthogonal to one which is odd in y.

3. Symmetric Differential Operators

Now that the boundary conditions have been reinterpreted as symmetry conditions across the boundary, it is time to introduce the Differential Operators that will transform these symmetries.

$$\Theta_s = \left(\frac{\partial^3}{\partial y^3} - 3\frac{\partial^3}{\partial y \partial x^2}\right) \tag{3}$$

$$\Theta_b = \left(\frac{\partial^3}{\partial x^3} - 3\frac{\partial^3}{\partial y^2 \partial x}\right). \tag{4}$$

These operators are closely related, and can also be identified as follows:

$$\left(\frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}\right)^3 = \Theta_b - \mathrm{i}\Theta_s. \tag{5}$$

To see how these operators transform the various symmetries we have discussed, consider the following results. Recall that our boundary is symmetric in the x axis, so any solution can be decomposed into solutions which are even or odd in y.

Theorem 2 *Let f be a solution to either the Dirichlet or Neumann Problem which is even (or odd) in y. Then $\Theta_s f$ is an orthogonal solution with the same eigenvalue and boundary condition.*

Proof: Note that the operator Θ_s has an odd number of partial derivatives with respect to y, so the symmetry in y will switch making the transformed solution orthogonal to the given one. The linearity of ∇^2 and Θ_s, along with using Clairaut's

Theorem to commute the partial derivatives, shows that $\Theta_s f$ has the same eigenvalue as f. The final thing to show is that the boundary conditions are satisfied. We will do this by demonstrating an alternate formulation of Θ_s. Consider $\frac{\partial^3}{\partial y^3}$ as the third directional derivative in the y-direction. Add to this the analogous third directional derivatives parallel to the other two sides and an interesting relationship emerges:

$$\left(\frac{\partial}{\partial y}\right)^3 + \left(\frac{\sqrt{3}}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial y}\right)^3 + \left(-\frac{\sqrt{3}}{2}\frac{\partial}{\partial x} - \frac{1}{2}\frac{\partial}{\partial y}\right)^3 = \frac{3}{4}\Theta_s. \tag{6}$$

By Theorem 1, the symmetry of f across the boundary is equivalent to the boundary condition (Dirichlet is Odd, Neumann is Even). Because of the symmetric way in which Θ_s was constructed, expanding Θ_s in local coordinates we see that there are an even number of partial derivatives with respect to the normal direction (ν), so the symmetry (and therefore the boundary condition) is preserved. ■

Note that it is possible that $\Theta_s f$ is trivial, which is necessarily the case when f is non-degenerate. Additionally, since the triangle can be used to tessellate the plane under reflections, each solution admits a family of solutions obtained through a linear transformation of the coordinates. We will refer to these related solutions as harmonics.

Corollary 3 *For a solution f to either the Dirichlet or Neumann Problem, if $\Theta_s f = 0$ then f is either the ground state or one of its harmonics.*

Proof: By Courant's Nodal Line Theorem [4], we know that the ground state must be non-degenerate. From Theorem 2 we must have that $\Theta_s f = 0$. Furthermore, since $\Theta_s f = 0$ is a local condition, we would expect all the harmonics of the ground state to vanish as well. Indeed, using this additional constraint, we can now solve for the ground state and its harmonics directly. Let f be a non-degenerate solution which satisfies the equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) f(x,y) = 0 \tag{7}$$

$$\left(\frac{\partial^3}{\partial y^3} - 3\frac{\partial^3}{\partial y \partial x^2}\right) f(x,y) = 0 \tag{8}$$

when $(x, y) \in int(\Delta)$. Using Equation 7 we can eliminate partial derivatives with respect to x from Equation 8 to obtain

$$0 = \left(4\frac{\partial^3}{\partial y^3} + 3k^2\frac{\partial}{\partial y}\right) f(x,y) \tag{9}$$

Treating this as an ODE and assuming a linear combination of separable solutions, one can use elementary techniques to find the general form of the solution. Fitting the boundary conditions yields a novel derivation of these special solutions which agree with previously published solutions ([5], [12], [13]). ∎

The operator Θ_s has now been demonstrated to transform the symmetry in y, so the resulting solution will be orthogonal to the given one. Next consider the other symmetric differential operator, Θ_b.

Theorem 4 *Let f be a solution to the Helmholtz equation.*

1. *If f is a solution to the Dirichlet Problem, then $\Theta_b f$ is a solution to the Neumann Problem.*

2. *If f is a solution to the Neumann Problem, then $\Theta_b f$ is a solution to the Dirichlet Problem.*

Proof: This proof is analogous to the one for Theorem 2 except we must think of Θ_b as being related to the sum of the third directional derivatives taken perpendicular to each side. This results in a change from one boundary condition to the other, but otherwise preserves the eigenvalue and symmetry of the solution. ∎

While these theorems demonstrate that Θ_s and Θ_b have great potential to shed light on these problems, we do not yet have confirmation that if we start with a non-trivial solution, the transformed solution is also non-trivial. To the contrary, we have seen in Corollary 3 there is a class of solutions which do become trivial.

In order to investigate this further, we will make use of the explicitly constructed solutions to the Dirichlet and Neumann Problems as they are presented in references [12] and [13]. For the Dirichlet Problem, we will denote the symmetric solutions by $D_1^{m,n}$ and the anti-symmetric solutions by $D_{-1}^{m,n}$. For the Neumann Problem, we will denote the symmetric solutions by $N_1^{m,n}$ and the anti-symmetric solutions by $N_{-1}^{m,n}$. It is shown in these articles that these solutions are non-trivial and degenerate for quantum numbers $n > m > 0$, and that the symmetric solutions are non-zero for the additional case where $m = n$. Additionally,

$$k^2 = \frac{4\pi^2}{27r^2}(m^2 + mn + n^2),$$

where r is the radius of the inscribed circle.

Theorem 5 *The following equations relate the solutions $D^{m,n}_{\{1,-1\}}$ and $N^{m,n}_{\{1,-1\}}$. For each equation, μ is either 1 or -1.*

$$\Theta_s D^{m,n}_{\mu} = \frac{4\pi^3\mu\sqrt{3}}{243r^3}\left[(n-m)(2m+n)(2n+m)\right] D^{m,n}_{-\mu} \tag{10}$$

$$\Theta_s N^{m,n}_{\mu} = \frac{4\pi^3\mu\sqrt{3}}{243r^3}\left[(n-m)(2m+n)(2n+m)\right] N^{m,n}_{-\mu} \tag{11}$$

$$\Theta_b D^{m,n}_{\mu} = \frac{4\pi^3}{27r^3}\left[m(m+n)n\right] N^{m,n}_{\mu} \tag{12}$$

$$\Theta_b N^{m,n}_{\mu} = -\frac{4\pi^3}{27r^3}\left[m(m+n)n\right] D^{m,n}_{\mu} \tag{13}$$

As such, any non-trivial solution transformed under the operators Θ_s and Θ_b remain non-trivial except under Θ_s when $m = n$. These exceptional cases are covered by Corollary (3).

Proof: Theorems 2 and 4 give us everything here except the coefficients. These are obtained through direct computation. We also make use of Theorem 8.1 from both [12] and [13]. ■

This theorem shows that Θ_b gives an explicit one-to-one correspondence between solutions to the Dirichlet Problem and Neumann Problem. Considering Θ_s as an operator from the solution space to itself, we have identified the kernel of this operator (in Corollary 3) as the ground state solution along with its harmonics. When the kernel is factored out, we are left with a graded involution on the remaining solutions which provides additional structure to the solution space.

4. Additional Comments

In this sections we explore the natural origin of the operators Θ_s and Θ_b, as well as a discussion about whether or not analogous operators may exist for other domains.

4.1. Origin of these Operators

In the context of the symmetry considerations above, it may be strange to some that the use of Representation Theory has not been deployed. Indeed, considering the action of the dihedral group of order 6 ($\mathcal{D}_6$) on the vector space of solutions to either the Dirichlet or Neumann Problem is rather useful. It is well known that there are 3 irreducible representations: two one dimensional ones (the trivial and sign representations), and a two dimensional representation. This two dimensional irreducible representation is clearly a place where solutions are degenerate. However, we have gone beyond this and shown that all solutions from the sign representation are also degenerate, a fact which representation theory alone does not anticipate.

It is also worth noting that the operators Θ_s and Θ_b are actually representations of $\mathcal{D}_6$. When acting on the ring $\mathbb{R}[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}]$, Θ_s is the lowest order representative of the sign representation. It is also interesting to note that the Lapalce Operator is the lowest non-trivial representative of the trivial representation, and that Θ_b is the second lowest non-trivial representative of the trivial representation. As such, we can see the operators Θ_s and Θ_b are unique in this respect.

4.2. Other Domains

The success of the operators Θ_s and Θ_b for the equilateral triangle relied heavily on various unique aspects of the equilateral triangle, and as such any analogous operators for other domains do not likely exist.

Imagining another domain where this might work we would first need a polygonal boundary in order to use Theorem 1. Secondly, the symmetry of the operator needed to match the symmetry of the domain so that Theorem 1 could be deployed to understand the transformed solution, which requires the domain to be a regular polygon. Further, in order to change the symmetry and create (non-zero) solutions, this symmetry must be odd. Even if we restrict ourselves to regular odd polygons, the question of analyticity still presents a problem. This was a necessary piece when verifying properties of the transformed boundary. To put to rest any hope that this could still work, it has been well established through precise numerical techniques that solutions of the sign representation in the regular polygon are often (and possibly always) non-degenerate and that the Neumann and Dirichlet Conditions do not admit solutions with equal energy. [7]

5. Conclusion

In this paper, we used these differential operators to exploit the internal and boundary symmetry of a solution to the Helmholtz equation within the equilateral triangle. Doing so provided unique insight within both the Dirichlet and Neumann Problems. Here, the differential operator Θ_s transformed a given solution to an orthogonal one. Then we showed that this transformed solution is only zero for the ground state and its harmonics, but otherwise was shown to be non-trivial.

Alternatively, the operator Θ_b transformed only the boundary condition but otherwise preserved the symmetry of the solution. It was shown that all non-trivial solutions transformed by Θ_b remained non-trivial, thereby constructing a one-to-one correspondence between the Dirichlet and Neumann Problem for the equilateral triangle.

This pair of operators exist for reasons unique to the Equilateral triangle, and give new understanding to the well-known solutions.

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