Using symmetry to generate solutions to the Helmholtz equation inside an equilateral triangle

Nathaniel Stambaugh and Mark Semon

Abstract: We consider solutions to the Helmholtz equation $\nabla^2\psi + k^2\psi = 0$ within an equilateral triangle which obey the Dirichlet conditions on the boundary. While the explicit solutions are well-known, we present new insight into the structure of these solutions through a novel perspective that exploits the symmetry of the boundary. An elementary result from representation theory is that a non-rotationally symmetric solution must be degenerate and one can orthogonalize the degenerate solutions using symmetry. We extend this result by developing a method that transforms any degenerate solution into an orthogonal solution. This method also gives increased insight into degenerate solutions which can then be used to better understand non-degenerate solutions (such as the ground state). We leverage this insight to present a novel derivation of all non-degenerate solutions. Finally, we establish two more general results. First, we construct a converse to the well known result that given solutions in the equilateral triangle, all solutions of the $(30^\circ, 60^\circ, 90^\circ)$ triangle can be obtained. Second, we present an explicit operator which transforms any solution that satisfies the Dirichlet conditions into a solution that satisfies the Neumann conditions, and *vice-versa*.

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1. Introduction.

The solutions to many important physical problems, such as electromagnetic waves in waveguides [1], lasing modes in nanostructures [2], the electronic structure of graphene [3] and the quantum eigenvalues and eigenfunctions for various potential energies [4] are obtained by solving the ubiquitous Helmholtz equation

$$\nabla^2\psi + k^2\psi = 0. \tag{1}$$

In this paper we discuss the solutions to this equation when the region of interest is an equilateral triangle (Δ) and when the solutions vanish on the boundary (i.e. when they satisfy the Dirichlet condition $\psi\big|_{\partial\Delta} = 0$). Although the explicit solutions in this case are well-known, ([2],[4],[5],[6],[7]) we present an alternative method of understanding them that does not involve solving the differential equation directly, but rather uses only symmetry arguments or a differential operator derived from

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symmetry considerations to generate new solutions from any given solution. Our method is based upon first showing that each solution within the equilateral triangle is a member of one of four symmetry classes, and then introducing symmetry operators and a differential operator which transform solutions in one symmetry class into those of another or from one value of k^2 to another. We note that since the methods used are based solely on the symmetry of the boundary they could shed light on the solutions in other symmetric domains which are not as well understood as those within the equilateral triangle.

Obviously any method that generates one or more new solutions to the Helmholtz equation from a given solution is quite a powerful and useful tool. The fact that relationships between solutions are found from symmetry transformations alone rather than by solving the Helmholtz equation directly makes the method even more attractive. The method also has the advantage of being able to produce solutions with prescribed symmetries, which can be important if the desired solution needs, or is known to possess, certain symmetries.

We end by relating solutions which satisfy the Dirichlet boundary conditions to those which satisfy the Neumann boundary conditions. In particular, we present a novel and constructive demonstration of the fact that the corresponding solutions have the same values for k^2 ([8], [9]). This also gives a concrete and non-trivial example of the Dirichlet to Neumann map.

The paper is structured as follows: in Section 2 we establish our notation while reviewing the results from representation theory and linear algebra which are used in the rest of the paper. In Section 3 we show that every solution to the Helmholtz equation within an equilateral triangle is a member of one of four symmetry classes. In Section 4 we show how to take a solution from any one of the four classes and generate from it solutions in a different symmetry class and/or with different values of the scalar k^2. In Section 5 we use our approach to present a novel derivation of the explicit solutions to the Helmholtz equation when the solution is non-degenerate. This includes the solution for the lowest value of k^2, i.e. the "ground state solution." We show in Section 6 that a similar operator to one developed in Section 4.3 can be used to transform solutions that obey the Dirichlet boundary conditions into solutions that obey the Neumann boundary conditions and vice versa. In Section 7 we summarize our results and discuss the various ways in which they can be applied. In particular, we discuss the correspondence between solutions in the equilateral and $(30^\circ, 60^\circ, 90^\circ)$ triangles.

2. Notation and Background

2.1. Representation Theory

A representation is a homomorphism ρ from the group G into the group of linear transformations of a vector space (in our case, the real numbers suffice), which we denote by $\rho : G \rightarrow \mathrm{GL}_n(\mathbb{R})$. The representation assigns to each group element a transformation of the vector space that is consistent with the multiplication table of the group. For the dihedral group $\mathcal{D}_3$, every such representation can be decomposed into a direct product of three irreducible representations.

Before we can describe these homomorphisms we need to describe the elements of the group $\mathcal{D}_3$. Let σ be a 120° counter-clockwise rotation about the center of an equilateral triangle and μ be a reflection (without loss of generality) about the $x-$axis, as shown in Figure (1). The defining relationship of the dihedral group says that $\mu\sigma = \sigma^{-1}\mu$. Since these two elements generate the whole group, we need only define each homomorphism on these generators. Listing the elements of the group, we have

$$\mathcal{D}_3 = \{e, \sigma, \sigma^2, \mu, \mu\sigma, \mu\sigma^2\}.$$

The first irreducible representation is called the *trivial* representation because it maps every group element to the identity map of $\mathbb{R}$. While this may seem somewhat, well, trivial, it actually plays an interesting role later on. Symbolically, $\rho_1(\alpha) = 1$ for every $\alpha \in \mathcal{D}_3$.

The second representation is called the *sign* representation, though it also could be called the *orientation* representation, because it shows whether a reflection has occurred. That is, $\rho_2(\sigma) = 1$ and

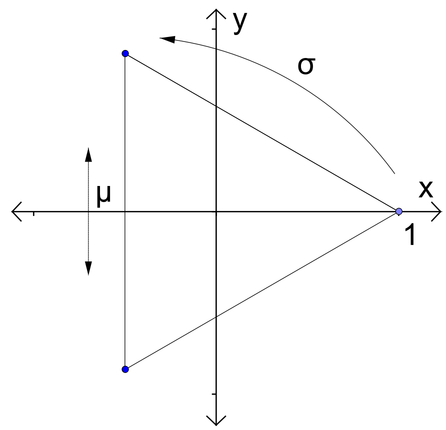


Fig. 1. The generators of the symmetry group for the equilateral triangle. We let σ denote a counter-clockwise rotation by 120° and μ the reflection in the x-axis.

$\rho_2(\mu) = -1$. Obviously the trivial and sign representations are one dimensional.

The third representation is the only one that displays every nuance in the group, and it therefore is sometimes used to define $\mathcal{D}_3$. Unlike the two previous representations, ρ_3 is a two dimensional representation whose elements (in $\mathrm{GL}_2(\mathbb{R})$) are

$$\rho_3(\sigma) = \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, \qquad \rho_3(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Using ρ_3 we can define in a natural way the action of each group element α on a solution $f(x, y)$ of the Helmholtz equation . The argument of the function is the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$, so the following action is well defined:

$$(\alpha \cdot f)(x, y) = f\Big(\rho_3(\alpha)^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\Big). \tag{2}$$

The use of the inverse of the representation matrix is required to make this a homomorphism, and can be thought of as a passive transformation on the coordinates. In order to simplify the notation, we will write αf in place of $\alpha \cdot f$.

2.2. Inner Product Spaces

If f_1 and f_2 are two solutions of the Helmholtz equation then we define the inner product of f_1 and f_2 as

$$\langle f_1, f_2 \rangle = \int\int_\Delta f_1(x, y) f_2(x, y) dx dy, \tag{3}$$

where the integral is taken over the domain Δ. The norm (or length) of a solution f is defined as

$$||f|| = \sqrt{\langle f, f \rangle}. \tag{4}$$

This is called the $\mathcal{L}^2$ norm, which we will use to normalize any given solution and also to establish when two solutions f_1 and f_2 are orthogonal ($\langle f_1, f_2 \rangle = 0 \iff f_1 \perp f_2$).

If two solutions have the same value of k and are orthogonal, we can form a two dimensional space spanned by these solutions. In the same way that we use the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ to represent $a\hat{i} + b\hat{j}$, in our context this column vector will represent $af_1 + bf_2$.

3. Classifying solutions to the Helmholtz equation by their symmetries.

If f is a solution of the Helmholtz equation within a region (denoted by Δ) whose sides form an equilateral triangle, and which satisfies the Dirichlet conditions on the boundary, then for each element $\alpha \in D_3$, αf (as defined by Eq. (2)) is a solution of the Helmholtz equation with the same value of k^2. In this section we prove that every such solution f belongs to one of four sets according to its rotational and reflection symmetries. We call these sets *symmetry classes* and denote them by A1, A2, E1 and E2.

We first consider the rotational symmetries of solutions of the Helmholtz equation. If a solution f is rotated to obtain a new solution σf then, in general, the new solution can be rotationally symmetric ($\sigma f = f$), rotationally anti-symmetric ($\sigma f = -f$), or rotationally asymmetric ($\sigma f \neq \pm f$). We can eliminate the rotationally anti-symmetric case as follows: Suppose that $\sigma f = -f$. Then, since σ^3 is the identity element,

$$f = \sigma^3 f = \sigma\sigma(-f) = \sigma f = -f. \tag{5}$$

Thus $f(x) = -f(x)$ for every $x \in \Delta$ so f is identically zero on the domain. Therefore, when we rotate a (non-trivial) solution f to obtain a new solution σf, the new solution σf must be either rotationally symmetric or rotationally asymmetric. In what follows we first consider the effect of reflections on the rotationally symmetric solutions and then the effect of reflections on the rotationally asymmetric solutions.

3.1. Properties of rotationally symmetric solutions under reflection.

Assume that f is a rotationally symmetric solution of the Helmholtz equation, so $\sigma f = f$. In general, the solution μf can be symmetric, anti-symmetric, or asymmetric under reflection. However, we can eliminate the asymmetric case as follows: Suppose $\mu f \neq \pm f$. Define the two functions

$$f_+ = \tfrac{1}{2}(f + \mu f), \tag{6}$$
$$f_- = \tfrac{1}{2}(f - \mu f). \tag{7}$$

The functions f_+ and f_- are solutions of the Helmholtz equation because each is constructed from a linear combination of solutions to the Helmholtz equation. Furthermore, f_+ is symmetric and f_- is anti-symmetric under reflections about the $x-$axis, and the solution f can be written as $f = f_+ + f_-$. In addition, the boundary condition obeyed by f will also be obeyed by f_+ and f_-. Consequently, any asymmetric solution f can be decomposed into the sum of the symmetric and anti-symmetric solutions f_+ and f_-, and we thus need only consider solutions to the Helmholtz equation which are symmetric or anti-symmetric under reflection about the x-axis.

If we denote by A1 the set of solutions that are symmetric under a rotation σ and symmetric under a reflection μ, and by A2 the set of solutions that is symmetric under a rotation σ and anti-symmetric under reflection μ , then we can summarize the results of this section with the following table:

Table 1. Rotationally Symmetric Solutions

	σf_i	μf_i
$f_1 \in$A1	$+f_1$	$+f_1$
$f_2 \in$A2	$+f_2$	$-f_2$

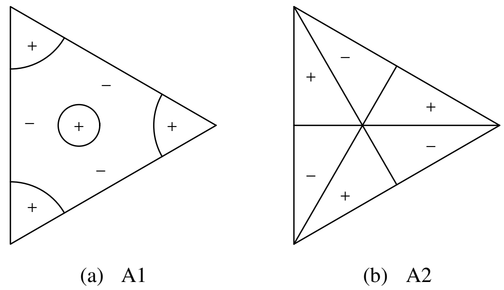


Fig. 2. Figures (a) and (b) show solutions f which both have rotational symmetry. The solution shown in (a) is also symmetric under reflection, while the solution shown in (b) is also anti-symmetric under reflections.

Figure (2) shows examples of solutions in the symmetry classes A1 and A2.

It's interesting to note that all of the solutions in A1 are orthogonal to all the solutions in A2: Suppose $f_1 \in A1$, then $\mu f_1 = f_1 \iff f_1(x,-y) = f_1(x,y)$, which means that if $f_1 \in A1$ then f_1 is even in y. Similarly, if $f_2 \in A2$, then $\mu f_2 = -f_2 \iff f_2(x,-y) = -f_2(x,y)$, which means that if $f_2 \in A2$ then f_2 is odd in y. Therefore the product $f_1 f_2$ is odd in $y \Rightarrow \langle f_1, f_2 \rangle = 0 \iff f_1 \perp f_2$

We end this subsection by noting that $f_1 \in$ A1 is characterized in the trivial representation by

$$\alpha f_1 = \rho_1(\alpha) f_1, \tag{8}$$

and $f_2 \in$A2 is characterized in the sign representation

$$\alpha f_2 = \rho_2(\alpha) f_2. \tag{9}$$

3.2. Properties of rotationally asymmetric solutions under reflection.

As was shown at the beginning of Section 3, if f is a solution of the Helmholtz equation in the region Δ then the rotated solution σf will be either rotationally symmetric or rotationally asymmetric. Having examined the reflection properties of the rotationally symmetric solutions in the previous section we now examine the reflection properties of the rotationally asymmetric solutions. Although the lack of rotational symmetry might make us expect that these solutions will be of little use, on the contrary, not only are they the most common solutions to the Helmholtz equation, but they also have a number of interesting and useful properties.

Let E1 be the set of solutions which are asymmetric under rotations and symmetric under reflections, and E2 be the set of solutions which are asymmetric under rotations and anti-symmetric under reflections. Consider a normalized solution f_1 in class E1. At this point we do not know anything about σf_1 except that it is a solution with the same value of k^2 as f_1. For that matter, so is $\sigma^2 f_1$. So consider the function $\hat{f}_2 = \sigma f_1 - \sigma^2 f_1$. We now show that $\hat{f}_2$ is in symmetry class E2:

$$\begin{aligned} \mu \hat{f}_2 &= \mu(\sigma f_1 - \sigma^2 f_1) \\ &= \mu\sigma f_1 - \mu\sigma^2 f_1 \\ &= \sigma^2 \mu f_1 - \sigma\mu f_1 \\ &= \sigma^2 f_1 - \sigma f_1 \\ &= -\hat{f}_2 \end{aligned} \tag{10}$$

The reason we call this new solution $\hat{f}_2$ is that it is not normalized; we will call the normalized function f_2. We also note that f_1 and f_2 are orthogonal since their product is odd in y.

Since the action of the group introduces a second solution with the same value of k^2, we consider the two dimensional solution space spanned by f_1 and f_2. As in Section 2(a), the vector $\vec{f} = \begin{bmatrix} a \\ b \end{bmatrix}$ represents the solution $f = af_1 + bf_2$. Written in this way, we can recognize the third irreducible representation ρ_3

$$\alpha \vec{f} = \rho_3(\alpha)\vec{f}.$$

This equation actually contains a lot of information, and is strikingly similar to Eqs.(8) and (9). We now use it to normalize $\hat{f}_2$.

Let $\vec{f_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then σf_1 can be re-expressed as a linear combination of f_1 and f_2 as follows:

$$\begin{aligned} \sigma f_1 &= \frac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} && (11) \\ &= \frac{1}{2}\begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix} \\ &= -\frac{1}{2}f_1 + \frac{\sqrt{3}}{2}f_2. && (12) \end{aligned}$$

Similarly, $\sigma^2 f_1 = -\frac{1}{2}f_1 - \frac{\sqrt{3}}{2}f_2$. Therefore

$$\sigma f_1 - \sigma^2 f_1 = \sqrt{3}f_2, \tag{13}$$

and solving for f_2, we find the normalized solution

$$f_2 = \frac{1}{\sqrt{3}}(\sigma f_1 - \sigma^2 f_1). \tag{14}$$

Similarly, f_1 can be expressed in terms of f_2 as

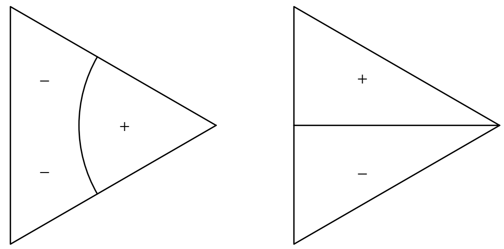
$$f_1 = \frac{1}{\sqrt{3}}(\sigma f_2 - \sigma^2 f_2). \tag{15}$$

Consequently, given any solution f_1 in the symmetry class E1 we can generate from it an orthogonal solution f_2 in the symmetry class E2, and *vice versa*. Figure 3 shows two examples of solutions from E1 and E2.

To summarize, in this section we have established the important result that every solution to the Helmholtz equation within an equilateral triangle can be placed into one of four symmetry classes. The class A1 is the set of solutions which are symmetric under rotation and under reflection, the class A2 is the set of solutions which are symmetric under rotation and anti-symmetric under reflection, the class E1 is the set of solutions which are asymmetric under rotation and symmetric under reflection, and the class E2 is the set of solutions which are asymmetric under rotation and anti-symmetric under reflection. These results are summarized in Table 2.

4. Generating new solutions from a given solution.

In the previous section we showed how Eq.(15) can be used to generate a solution in the symmetry class E2 (i.e. a solution that is anti-symmetric when reflected about the x-axis) from any solution in



(a) E1 (b) E2

Fig. 3. Figures (a) and (b) show two solutions f, neither of which has rotational symmetry. The solution f_1 in (a) is symmetric under reflection and in the set E1, while the solution f_2 in (b) is anti-symmetric under reflection and in the set E2.

Table 2. The four symmetry classes

	Rotation	
Reflection	Symmetric	Asymmetric
Symmetric	A1	E1
Anti-Symmetric	A2	E2

the symmetry class E1 (i.e. from a solution that is symmetric when reflected about the x-axis) and *vice versa* (using Eq.(14)). More specifically, if we have a solution that is even in the y-coordinate and asymmetric under rotations σ, then we can generate from it a solution that is odd in the y-coordinate and asymmetric under rotations σ, and *vice versa*. Furthermore, in each case, the generated solution is orthogonal to the original solution. In this section we show when it is possible to take a solution from one of the four symmetry classes and generate from it an orthogonal solution in one of the other symmetry classes and/or with a different value of k^2. In fact, we present three ways to generate new solutions from a given solution.

First, we show how to build a many-to-one correspondence between solutions in symmetry class A2 and symmetry class E2. Next we present a way to take *any* solution and generate from it a solution with a larger scalar, which we will call a "harmonic" of the original solution. The final technique introduces a novel differential operator which transforms a solution from symmetry class A2 into a (non-trivial) solution in symmetry class A1. Applying this same technique to a solution in symmetry class A1 yields a solution in symmetry class A2. The resulting solution in A2 may be the trivial solution, and we discuss how this fact gives new insight into the "ground state solution."

We begin by quoting a theorem usually attributed to Lamé, using the statement (and referring the reader to the proof) given by McCartin [7]:

Theorem 1 *(Lamé) Suppose that $T(x,y)$ is a solution to the Helmholtz equation which can be represented by the trigonometric series*

$$T(x,y) = \sum_i \left(A_i \sin(\lambda_i x + \mu_i y + \alpha_i) + B_i \cos(\lambda_i x + \mu_i y + \beta_i)\right), \tag{16}$$

with $\lambda_i^2 + \mu_i^2 = k^2$. Then

1. *$T(x,y)$ is antisymmetric about any line along which it vanishes;*
2. *$T(x,y)$ is symmetric about any line along which its normal derivative, $\frac{\partial T}{\partial \nu}$, vanishes.*

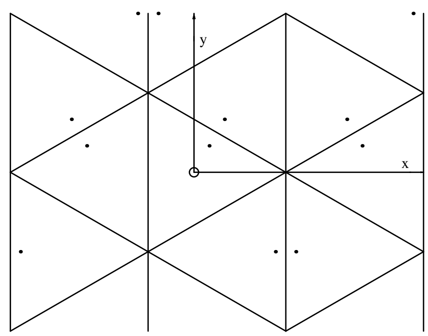


Fig. 4. This shows how the fundamental domain tiles the plane via reflections. Note how the location of the point in the fundamental domain changes as the reflections are made.

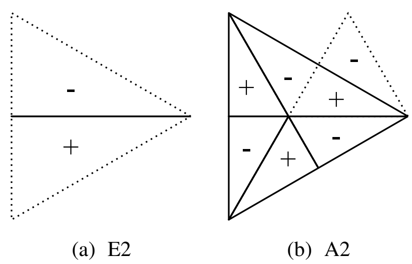


Fig. 5. A pictoral representation of solutions in the symmetry classes E2 and A2.

Lamé [7] also proves that the solutions to the Helmholtz equation in a triangular region subject to the Dirichlet conditions can be expressed in this way and that they form a complete, orthonormal set. Explicit expressions for these solutions are also given by Doncheski *et al.* [4]

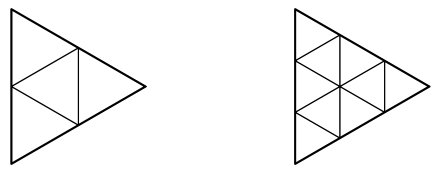
4.1. E2 $\leftrightarrow$ A2

In order to relate solutions in the symmetry classes E2 and A2 we use a method called "tessellating the plane," which extends any solution within the triangular domain to the plane. We begin the tessellation by defining the triangular region in which we are working as the "fundamental domain" and then we reflect this domain across each of its three boundaries. An example of this construction is shown in Figure (4). Tessellating the plane provides a way to smoothly extend the solution from the triangular region of interest to the plane.

We begin by considering a solution f_2 in the symmetry class E2. Since f_2 is antisymmetric under reflection it must be zero along the x axis. Thus, any solution in the symmetry class E2 will have the form shown in Figure (5a), with possibly more nodal curves. Similarly, if a solution is in the symmetry class A2, it not only needs a nodal line along the x-axis, but also along the other altitudes. Thus, any solution in the symmetry class A2 will have the form shown in Figure (5b), with possibly more nodal curves.

By examining the equilateral triangles in Figure (5) we see that the linear transformation

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 3x + \sqrt{3}y + 3 \\ -\sqrt{3}x + 3y + \sqrt{3} \end{bmatrix} \tag{17}$$



(a) $n = 2$ (b) $n = 3$

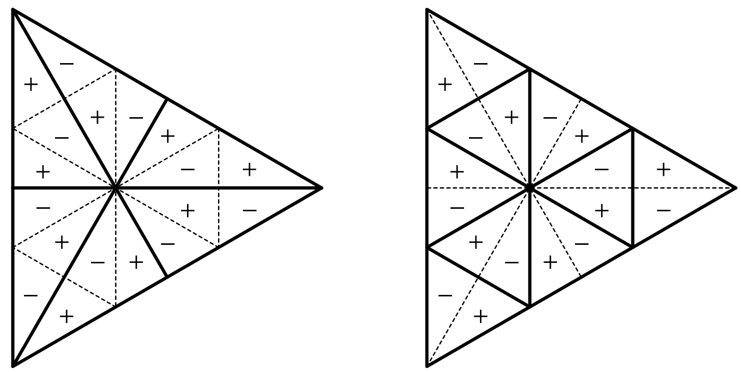
Fig. 6. The equilateral triangle can be decomposed into n^2 equilateral triangles

maps the dashed region in Figure (5a) into the dashed region in Figure (5b) [The origin is ℓ units from each vertex]. The A2 solution can then be constructed by tessellating the dashed triangle in Figure (5b) and thus creating the triangle in that figure who boundaries are shown with the solid lines. Therefore, the transformation in Eq. (17), which is deduced from the figures, takes *any* solution in the E2 symmetry class and transforms it into a solution in the A2 symmetry class. Having determined the explicit form of the transformation from the tessellation we can rewrite the Helmholtz equation in the new coordinates and deduce the new value of the scalar term. In this way we can construct an explicit solution in A2 from any given solution in E2 and determine the scalar term associated with it. We carry out this calculation explicitly in Appendix I and show that if the original function in E2 is a solution of the Helmholtz equation with the scalar k^2, then the solution in A2 generated from it will be a solution of the Helmholtz equation with the scalar $3k^2$. Similarly, if we start with a solution in A2 (rather than E2) of the Helmholtz equation with the scalar k^2, then the same argument will produce another solution in A2 but with the scalar $3k^2$. In summary if we start with any function in A2 or E2 that is a solution to the Helmholtz equations with the scalar k^2 then, using the method presented in Appendix I, we can generate a function in A2 that is a solution to the Helmholtz equation with the scalar $3k^2$.

The proof given in Appendix I is reversible, so if we start at the end of the argument with a function in A2 which is a solution of the Helmholtz equation with scalar k^2, then we can run the argument backwards and generate another solution which is in either A2 or E2. In both cases the function produced will be a solution of the Helmholtz equation with the scalar $\frac{k^2}{3}$. However, since there exists a solution with the lowest allowed value of k^2 (the "ground state solution") the process of generating solutions with smaller values of the scalar must stop. Note that this method establishes a many-to-one correspondence between solutions in symmetry class A2 and symmetry class E2.

4.2. Generating solutions with higher values of k^2

Although we can use the proof presented in the Appendix I to generate from a solution in A2 with the scalar k^2 new solutions in A2 or E2 with three times the value of the scalar k^2, there is a more direct way to take a solution and generate from it solutions with higher values of the scalar for solutions from *any* symmetry class. To do this we carry out a different tessellation of the plane and then extract the new value of the scalar from the coordinate transformation. The tessellation, which is shown in Figure (7), decomposes the equilateral triangle into n^2 equilateral sub-triangles, for any $n \in \mathbb{N}$. As we show in Appendix II, the explicit coordinate transformation that creates the sub-triangles is constructed from a dilation followed by a translation along the x-axis. Carrying out this transformation we can start with a function in any symmetry class that is a solution of the Helmholtz equation with the scalar k^2 and generate from it a family of new solutions to the Helmholtz equation with scalar n^2k^2. Note that if the original solution is in the symmetry class E2 (i.e. is rotationally asymmetric) then this approach will produce a solution in A2 (i.e. the generated solution will become rotationally symmetric) whenever n is divisible by 3. In this case, however, we have would have already constructed this solution by using



(a) Using the method from Section 4.1 twice.

(b) Harmonic with $n = 3$.

Fig. 7. These pictures show two redundant constructions of a higher energy state.

the method from Section (4.1) twice. See Figure 7 for a schematic proof of this fact.

Note that any solution with a vertical nodal line can be reduced using this method. Extending the solution to the plane, we can use Theorem 1 and any vertical nodal line to introduce a new mirror symmetry. Combining this with the mirror symmetries about the boundaries and the rotational symmetry, there will be a fundamental domain without vertical nodal lines. This represents a solution which can be used to generate the original solution without any vertical nodal lines.

4.3. Generating even solutions from odd solutions and *vice versa*.

In this section we prove that there exists a differential operator that transforms a solution f_2 in A2, i.e. a solution that is symmetric under rotations and an odd function of y, into a solution f_1 in A1, that is, into a solution which is symmetric under rotations and an even function of y. To show this we construct a function $\hat{f}_1$ in A1 (where f_1 is the normalized solution) by defining $\hat{f}_1$ in the following way:

$$\hat{f}_1 = \left(\frac{\partial^3}{\partial y^3} - 3\frac{\partial^3}{\partial y \partial x^2}\right) f_2(x, y). \tag{18}$$

The proof that $\hat{f}_1$ is in A1 goes as follows: First, it is easy to see that $\hat{f}_1$ is a solution to the Helmholtz equation with the scalar k^2 *iff* f_2 is a solution with the same scalar k^2. To complete the proof we need to show four more things: first, that $\hat{f}_1$ is rotationally symmetric, second, that $\hat{f}_1$ satisfies the Dirichlet conditions, third, that $\hat{f}_1$ is even in the y-coordinate, and fourth, that $\hat{f}_1$ is not identically zero.

With single variable functions, one common way of generating an even function from an odd function is to take the first derivative. However, the need to satisfy all of the above conditions requires a more complicated procedure. The generalization of the first derivative in one dimension to two dimensions is the directional derivative, and the directional derivative of a function f in the direction $\hat{e}$ is $\nabla f \cdot \hat{e}$. Higher order directional derivatives are simply powers of this operator. As can be easily verified, although the first directional derivative of f_2 satisfies the Helmholtz equation it would not necessarily have the correct rotational symmetry. To correct this, we can *symmetrize* the solution by adding it to both of its rotates. The resulting solution will now be even and rotationally symmetric. Additionally, the transformed solution will satisfy the boundary (Dirichlet) condition since any nodal line parallel to a side will remain a nodal line. This follows from Theorem 1, since every solution is antisymmetric in a nodal line. Thus a directional derivative along the nodal line is zero, and by anti-symmetry the other two directional derivatives will cancel out. Unfortunately, this particular method always yields the trivial solution, so it is not very helpful. Indeed, we can imagine the directional derivatives in the tangent plane, and by symmetry they will always add to zero.

However, this leads us to try the next odd power of the directional derivative in the y-direction, $\frac{\partial^3}{\partial y^3}$. Once again, the required rotational symmetry leads us to symmetrize the solution by adding to the third directional derivative in the y-direction the third directional derivative in the directions parallel to the other two sides. We can describe the resulting differential operator algebraically by rotating the directional derivative by σ:

$$\begin{aligned} & \left(\nabla \cdot \hat{j}\right)^3 + \left(\nabla \cdot (\sigma \cdot \hat{j})\right)^3 + \left(\nabla \cdot (\sigma^2 \cdot \hat{j})\right)^3 \\ = & \left(\frac{\partial}{\partial y}\right)^3 + \frac{1}{8}\left(\sqrt{3}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^3 + \frac{1}{8}\left(-\sqrt{3}\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^3 \\ = & \frac{3}{4}\left(\frac{\partial^3}{\partial y^3} - 3\frac{\partial^3}{\partial y \partial x^2}\right) \end{aligned} \tag{19}$$

This gives us our differential operator from Eq. (18), apart from a multiplicative constant (which is unimportant since the resulting solution will still need to be normalized). With this new understanding of Eq. (18), we can then argue as before that the new function $\hat{f}_1$ has the correct symmetry and satisfies the Dirichlet conditions. Using the Helmholtz relation, note that

$$\frac{\partial^3}{\partial y \partial x^2} = -\frac{\partial^3}{\partial y^3} - k^2 \frac{\partial}{\partial y}. \tag{20}$$

Combining this with our differential operator, we can rewrite it as

$$4\frac{\partial^3}{\partial y^3} + 3k^2 \frac{\partial}{\partial y}. \tag{21}$$

Written in this way, it is clear the $\hat{f}_1$ is now symmetric in the x-axis. The last thing we need to do is show that the solution is non-zero. Without loss of generality we can assume there are no vertical nodal lines on the interior of the triangle. (This is possible using the last paragraph from Section 4.2.)

Since any solution will be analytic on the interior of the triangle, we consider the power series of the function $f_2(x, y)$ at the origin.

$$f_2(x, y) = \sum_{i,j} c_{i,j} x^i y^j \tag{22}$$

Consider the solution along the y-axis, $f_2(0, y)$. By symmetry, we know that there are only odd terms. Additionally, note that the directional derivatives at the origin parallel to each side are all equal, also by symmetry. We have previously argued that the sum of the three directional derivatives with rotational symmetry gives the zero function, thus $c_{0,1} = 0$.

$$f_2(0, y) = f_2(y) = \sum_{j \geq 3,\ \text{odd}} c_{0,j} y^j \tag{23}$$

By assumption, $f_2(0,y)$ is not identically zero, so there is a non-zero term with minimal index. Now consider $\hat{f}_1$:

$$\begin{aligned}\hat{f}_1 &= \left(4\frac{\partial^3}{\partial y^3} + 3k^2\frac{\partial}{\partial y}\right) f_2(y) \\ &= 4f_2'''(y) + 3k^2 f_2'(y)\end{aligned} \tag{24}$$

Using the first non-zero term in the power series for f_2, we note that its third derivative is non-zero, and higher order terms cannot cancel it. Thus $\hat{f}_1$ is non-zero, so we can normalize it to get a new solution f_1. In addition, we can note that f_1 is non-zero almost everywhere. Indeed, if f_1 is zero on an open set, then by analyticity it would be zero everywhere.

While we have only shown that this process works for functions in A2 without a node along the y-axis, the strength of our conclusion shows that this is a local property. Combining this with earlier methods, the requirement to be non-zero along the y-axis can be eliminated.

Note that the argument that the transformed solution is nonzero did not use any facts about the boundary conditions. Later we will need this fact when working with solutions which satisfy the Neumann boundary conditions.

It should also be noted that the only place we used the fact that f_2 was anti-symmetric was to prove that f_1 was non-zero. Indeed, the process introduced in Eq. (18) can also be used to transform a symmetric solution into an anti-symmetric solution, though the transformed solution may be zero. In the next section we will show that if a solution in class A1 is transformed by this differential operator and becomes the zero function, then it was (a harmonic of) the "ground state." Otherwise, the transformed solution can be normalized to a solution in class A2, giving a one-to-one correspondence between solutions in class A2 and those solutions in A1 which are not harmonics of the ground state.

5. The Ground State Solution

In the previous section we introduced a differential operator (defined in Eq. 18) that transformed anti-symmetric solutions into symmetric solutions. One natural question is what does this operator do to the solution of the Helmholtz equation with the minimum value of k^2, that is, to the "ground state solution"? Since the ground state solution is always non-degenerate, the operator in Eq. (18) must transform it into the zero function. We will now show that any solution from class A1 which transforms to zero under this differential operator is the ground state or a harmonic of the ground state.

Let f be a solution in symmetry class A1 which satisfies the two equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k^2\right) f(x,y) = 0 \tag{25}$$

and

$$\left(\frac{\partial^3}{\partial y^3} - 3\frac{\partial^3}{\partial y \partial x^2}\right) f(x,y) = 0. \tag{26}$$

Using the first equation we can eliminate partial derivatives with respect to x from the second equation to obtain

$$0 = \left(4\frac{\partial^3}{\partial y^3} + 3k^2\frac{\partial}{\partial y}\right) f(x,y) \tag{27}$$

$$\Rightarrow 0 = \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial y^2} + \frac{3k^2}{4}\right) f(x,y). \tag{28}$$

Assuming a solution of the form $f(x,y) = \sum_i X_i(x)Y_i(y)$, then Equation 28 gives that $Y_1 = 1$, $Y_2 = \sin(\frac{\sqrt{3}k}{2}y)$ and $Y_3 = \cos(\frac{\sqrt{3}k}{2}y)$. However, since solutions in class A1 are symmetric in y, the only allowed solutions are Y_1 and Y_3.

Similarly, putting $f(x,y) = \sum X_i(x)Y_i(y)$ into Eq.(26), we get $X_1 = A_1\cos(kx) + B_1\sin(kx)$ and $X_3 = A_3\cos(kx/2) + B_3\sin(kx/2)$. Imposing the boundary condition at $x = -\ell/2$, we find that $X_1(x) = \sin(k(x+\ell/2))$ and $X_3(x) = \sin(k/2(x+\ell/2))$. Putting this all together we find the solution

$$f(x) = A\sin\left(k\Big(x+\frac{\ell}{2}\Big)\right) + B\sin\left(\frac{k}{2}\Big(x+\frac{\ell}{2}\Big)\right)\cos\left(\frac{\sqrt{3}k}{2}y\right).$$

Imposing the remaining boundary condition along $y = \frac{-x+\ell}{\sqrt{3}}$, we get

$$\begin{aligned} 0 &= A\sin\left(k\Big(x+\frac{\ell}{2}\Big)\right) + B\sin\left(\frac{k}{2}\Big(x+\frac{\ell}{2}\Big)\right)\cos\left(\frac{\sqrt{3}k}{2}\left(\frac{-x+\ell}{\sqrt{3}}\right)\right) \\ &= A\sin\left(k\Big(x+\frac{\ell}{2}\Big)\right) + B\sin\left(\frac{k}{2}\Big(x+\frac{\ell}{2}\Big)\right)\cos\left(\frac{k}{2}(x-\ell)\right) \\ &= A\sin\left(kx+\frac{k\ell}{2}\right) + \frac{B}{2}\left(\sin\left(\frac{3k\ell}{4}\right) + \sin\left(kx-\frac{k\ell}{4}\right)\right). \end{aligned}$$

This equation is satisfied when $B = \pm 2A$ and $\sin(3k\ell/4) = 0$, so $k = 4\pi n/3\ell$ for $n > 0$.

$$\begin{aligned} 0 &= \sin\left(kx+\frac{k\ell}{2}\right) \pm \sin\left(kx-\frac{k\ell}{4}\right) \\ &= \sin\left(kx+\frac{2\pi n}{3}\right) \pm \sin\left(kx-\frac{\pi n}{3}\right) \end{aligned}$$

Note that these two sine waves are horizontally shifted by πn, so there is a suitable choice of sign for B to make them cancel. Putting this all together,

$$f_n(x) = \sin\left(k_n\Big(x+\frac{\ell}{2}\Big)\right) + 2(-1)^n\sin\left(\frac{k_n}{2}\Big(x+\frac{\ell}{2}\Big)\right)\cos\left(\frac{\sqrt{3}\,k_n}{2}y\right), \quad (29)$$

where $k_n = \frac{4\pi n}{3\ell}$. For $n = 1$ this agrees with the accepted solution in the literature for the ground state solution with center-to-vertex length ℓ [4]. To compare Eq. (29) with the explicit solution given in McCartin, [7] note that if the inscribed circle has radius r then $k = \frac{2\pi}{3r}$, and if the side length is h then $k = \frac{4\pi\sqrt{3}}{3h}$. For larger values of n, we simply get the harmonics promised at the end of Section (4.3).

6. Dirichlet vs. Neumann Boundary Conditions

In the previous sections the solutions we considered were those satisfying the Dirichlet boundary conditions. In this section we consider solutions which satisfy the Neuman boundary conditions, that is, the solutions whose normal derivative vanishes on the boundary. The solutions to this problem are derived explicitly in [8], and some relationships between them are discussed in [9]. In this section we go beyond the work of McCartin1 [8] and Pinsky [9] by finding a differential operator, similar to the one derived in Section(4.3), which transforms solutions of the Dirichlet problem to solutions of the Neumann problem, and *vice versa*.

Note that Theorem 1 says that a line of symmetry implies that the Neumann condition is satisfied. Therefore we can modify the argument from Section (4.3) to transform the odd symmetry in the boundary (i.e. Dirichlet condition) to even symmetry in the boundary (i.e. Neumann condition). To do this, we take the third directional derivative in the x-direction, and add it to the rotates under 120°. Given the function f satisfying Dirichlet conditions, consider the function

$$F(x,y) = \left(\frac{\partial^3}{\partial x^3} - 3\frac{\partial^3}{\partial x \partial y^2}\right) f(x,y) \tag{30}$$

$$= \left(4\frac{\partial^3}{\partial x^3} + 3k^2\frac{\partial}{\partial x}\right) f(x,y) \tag{31}$$

By construction, the function F satisfies the Neumann conditions and is a solution to the Helmholtz equation with the same value of k^2 as f. The only thing we need to show is that the function F is not identically zero.

Consider the case where f is a nontrivial solution in symmetry class A1. Since any solution will be analytic on the interior of the triangle, we consider the power series of the function $f(x,y)$ at the origin, expanded in y. By symmetry, the only nonvanishing terms have even powers of y.

$$f(x,y) = \sum_{j \text{ even}} f_j(x) y^j \tag{32}$$

Next we will argue that $f_0(x)$ not an even function. Suppose for a contradiction that it is. Then rearranging the Helmholtz equation we get $f_{yy} = -f_{xx} - k^2 f(x)$, which we can interpret as an inductive statement that

$$f_{j+2} = \frac{-1}{(j+2)(j+1)}(f_j''(x) + k^2 f_j(x)). \tag{33}$$

Since we are assuming $f_0(x)$ is even, then by induction $f_j(x)$ is even for all j. Thus $f(x,y)$ is even in x. Since $f(x,y)$ vanishes along $x = -\ell/2$, it also vanishes along $x = \ell/2$. Following the nodal pattern in Figure 6(b) we find that $f(x,y)$ must also vanish along the line $x = 0$. By the theorem of Lamé, $f(x,y)$ is also odd in x. Thus f is the zero function, which gives us our contradiction.

Since we know that $f_0(x)$ is not even, the power series expansion for $f_0(x)$ has a non-zero odd term with minimal degree. Since $f(x,y)$ is rotationally symmetric at $(0,0)$, the directional derivatives perpendicular to the sides must all be equal. As we noted before, they also add to zero, so $f_0'(x) = 0$, so the minimal non-zero odd term has degree at least 3. Now consider F:

$$F(x,0) \quad = \quad 4f'''(x) + 3k^2 f'(x) \tag{34}$$

Using the first non-zero odd term in the power series for f, we note that its third derivative is non-zero, and higher order terms cannot cancel it. Thus $F(x,0)$ is non-zero, so $F(x,y)$ is not identically zero. Using transformations from Section 4, we can extend this result to all symmetry classes. In particular, note that a solution from symmetry class A2 can first be transformed to one in symmetry class A1, then the boundary conditions can be changed, then it can be transformed back to a solution in symmetry class A2 (since the solution is degenerate!).

Note that applying this operator to a solution with Neumann boundary conditions recovers the Dirichlet boundary conditions through a similar argument. Therefore this operator gives an explicit relationship between solutions to the Dirichlet problem and the Neumann problem, showing that their eigenvalues are equal as a result of symmetry considerations alone. In addition to the aforementioned relationship with the operator from Section 4, it is also interesting to consider the following third directional derivative:

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^3 f=\left(\frac{\partial^3}{\partial x^3}+3i\frac{\partial^3}{\partial x^2\partial y}-3\frac{\partial^3}{\partial x\partial y^2}-i\frac{\partial^3}{\partial y^3}\right)f \tag{35}$$

$$=\left(\frac{\partial^3}{\partial x^3}-3\frac{\partial^3}{\partial x\partial y^2}\right)f-i\left(\frac{\partial^3}{\partial y^3}-3\frac{\partial^3}{\partial x^2\partial y}\right)f \tag{36}$$

$$=F(x,y)-i\hat{f} \tag{37}$$

If f is a non-degenerate solution, then this shows that taking the given third directional derivative transforms the boundary from Dirichlet to Neumann conditions, and vice versa. For degenerate solutions, the real part of the transformed solution will have changed boundary conditions, while the imaginary part will have changed reflection symmetry and preserved rotational symmetry.

In light of this result, many of the earlier observations made for solutions to the Dirichlet problem can be extended to the Neumann problem. We leave these to the reader.

7. Conclusion

In this paper we have examined the solutions to the Helmholtz equation $\nabla^2\psi+k^2\psi=0$ within an equilateral triangle which obey the Dirichlet conditions on the boundary. We have shown that every solution is a member of one of four symmetry classes and that, from symmetry considerations alone, any given solution in one symmetry class can be used to generate solutions in another symmetry class and/or with other values of the scalar k^2. We also used symmetry considerations to find a novel derivation of the "ground state" solution of the Helmholtz equation and to transform the boundary conditions between Dirichlet and Neumann conditions.

These results have many interesting applications. For example, when collected together, the techniques in this article shed light on the relationship between solutions within the $(30°, 60°, 90°)$ and equilateral triangles. For example, solutions to the equilateral triangle which are in symmetry classes A2 and E2 vanish along the x-axis. If we restrict our domain Δ to quadrants I and II, these solutions become solutions to the $(30°, 60°, 90°)$ triangle (with the same value of k^2). See Figures 2(b) and 3(b), respectively.

Conversely, given a solution in the $(30°, 60°, 90°)$ triangle, we can reflect it across the x-axis to get a solution in the equilateral triangle that is in either of the symmetry classes A2 or E2. We can then take these solutions and construct from them solutions in A1 and E1 using the techniques from Section 4.3 and Section 3.2, respectively. Since both of these techniques are reversible, we know that *all* solutions within the equilateral triangle can be found from the solutions in the $(30°, 60°, 90°)$ triangle except for the ground state solution and its harmonics. However, these solutions can be derived directly using the technique in Section 4.3, which means that knowing all of the solutions within the $(30°, 60°, 90°)$ triangle enables us to obtain all of the solutions within the equilateral triangle.

In summary, we provided a new way to understand the relationship between degenerate solutions and exploited this relationship for non-degenerate solutions to give a novel derivation of these solutions explicitly. We introduced a related operator which makes explicit the relationship between solutions to the Dirichlet problem and Neumann Problem. Finally we established a converse to the well known method which constructs solutions tothe $(30°, 60°, 90°)$ triangle from particular solutions to the equilateral triangle. To do this we now better understand the two-to-one correspondence between the solutions in the equilateral triangle which are not harmonics of the ground state and the solutions of the $(30°, 60°, 90°)$ triangle.

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A. Appendix I

We want to show that the solution $g(x,y) = (f \circ T^{-1})(x,y) = f(X(x,y), Y(x,y))$ is a solution to the differential equation

$$\left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\right) g(x,y) = 3k^2 g(x,y)$$

when f is a solution to the differential equation

$$\left(\frac{\partial^2}{\partial^2 X} + \frac{\partial^2}{\partial^2 Y}\right) f(X,Y) = k^2 f(X,Y).$$

To make this a bit cleaner, we will need to write the transformation $T^{-1}(x,y)$ explicitly. Solving the system of equations

$$\begin{aligned} x &= \frac{1}{6}\left(3X + \sqrt{3}Y + 3\right) \\ y &= \frac{1}{6}\left(\sqrt{3}X - 3Y + \sqrt{3}\right) \end{aligned}$$

for X and Y, we get

$$\begin{aligned} X(x,y) &= \frac{1}{4}\left(6x - 2\sqrt{3}y + \sqrt{3} - 3\right) \\ Y(x,y) &= \frac{1}{4}\left(2\sqrt{3}x + 6y - \sqrt{3} - 3\right). \end{aligned}$$

In order to evaluate $\frac{\partial^2}{\partial^2 x} f(X,Y)$, we use the chain rule:

$$\begin{aligned}
\frac{\partial^2}{\partial^2 x} f(X,Y) &= \frac{\partial}{\partial x}\left(\frac{\partial X}{\partial x}\frac{\partial}{\partial X} f(X,Y) + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} f(X,Y)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{6}{4}\frac{\partial}{\partial X} f(X,Y) + \frac{2\sqrt{3}}{4}\frac{\partial}{\partial Y} f(X,Y)\right) \\
&= \frac{\partial}{\partial x}\left(\frac{3}{2} f_X(X,Y) + \frac{\sqrt{3}}{2} f_Y(X,Y)\right) \\
&= \frac{3}{2}\frac{\partial}{\partial x} f_X(X,Y) + \frac{\sqrt{3}}{2}\frac{\partial}{\partial x} f_Y(X,Y) \\
&= \frac{3}{2}\left(\frac{\partial X}{\partial x}\frac{\partial}{\partial X} f_X(X,Y) + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} f_X(X,Y)\right) + \\
&\qquad \frac{\sqrt{3}}{2}\left(\frac{\partial X}{\partial x}\frac{\partial}{\partial X} f_Y(X,Y) + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} f_Y(X,Y)\right) \\
&= \frac{3}{2}\left(\frac{3}{2} f_{XX}(X,Y) + \frac{\sqrt{3}}{2} f_{XY}(X,Y)\right) + \\
&\qquad \frac{\sqrt{3}}{2}\left(\frac{3}{2} f_{YX}(X,Y) + \frac{\sqrt{3}}{2} f_{YY}(X,Y)\right) \\
&= \frac{9}{4} f_{XX}(X,Y) + \frac{3\sqrt{3}}{2} f_{XY}(X,Y) + \frac{3}{4} f_{YY}(X,Y)
\end{aligned}$$

Similarly, for $\frac{\partial^2}{\partial^2 y} f(X,Y)$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial^2 y} f(X,Y) &= \frac{\partial}{\partial y}\left(\frac{\partial X}{\partial y}\frac{\partial}{\partial X} f(X,Y) + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} f(X,Y)\right) \\
&= \frac{\partial}{\partial x}\left(-\frac{2\sqrt{3}}{4}\frac{\partial}{\partial X} f(X,Y) + \frac{6}{4}\frac{\partial}{\partial Y} f(X,Y)\right) \\
&= \frac{\partial}{\partial y}\left(-\frac{\sqrt{3}}{2} f_X(X,Y) + \frac{3}{2} f_Y(X,Y)\right) \\
&= -\frac{\sqrt{3}}{2}\frac{\partial}{\partial y} f_X(X,Y) + \frac{3}{2}\frac{\partial}{\partial y} f_Y(X,Y) \\
&= -\frac{\sqrt{3}}{2}\left(\frac{\partial X}{\partial y}\frac{\partial}{\partial X} f_X(X,Y) + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} f_X(X,Y)\right) + \\
&\qquad \frac{3}{2}\left(\frac{\partial X}{\partial y}\frac{\partial}{\partial X} f_Y(X,Y) + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} f_Y(X,Y)\right) \\
&= -\frac{\sqrt{3}}{2}\left(-\frac{\sqrt{3}}{2} f_{XX}(X,Y) + \frac{3}{2} f_{XY}(X,Y)\right) + \\
&\qquad \frac{3}{2}\left(\frac{-\sqrt{3}}{2} f_{YX}(X,Y) + \frac{3}{2} f_{YY}(X,Y)\right) \\
&= \frac{3}{4} f_{XX}(X,Y) - \frac{3\sqrt{3}}{2} f_{XY}(X,Y) + \frac{9}{4} f_{YY}(X,Y)
\end{aligned}$$

Combining these two equations, we get

$$\begin{aligned}
\nabla^2 g(x,y) &= \frac{\partial^2}{\partial^2 x} f(X,Y) + \frac{\partial^2}{\partial^2 y} f(X,Y) \\
&= \frac{9}{4} f_{XX}(X,Y) + \frac{3\sqrt{3}}{2} f_{XY}(X,Y) + \frac{3}{4} f_{YY}(X,Y) \\
&\qquad \frac{3}{4} f_{XX}(X,Y) - \frac{3\sqrt{3}}{2} f_{XY}(X,Y) + \frac{9}{4} f_{YY}(X,Y) \\
&= 3(f_{XX}(X,Y) + f_{YY}(X,Y)) \\
&= 3k^2 f(X,Y) \\
&= 3k^2 g(x,y)
\end{aligned}$$

which is the desired result.

B. Appendix II

In this section we examine the properties of solutions obtained by de-composing the fundamental domain into n^2 equilateral triangles. First we show that if we start with a solution f in A2 or E2 which is a solution of the Helmholtz equation with the scalar k^2 then, by decomposing the triangular domain into n^2 equilateral triangles, we can generate a solution to the Helmholtz equation with scalar n^2k^2.

The transformation of the plane which replaces the original triangle by n^2 triangles is a pure dilation by a factor of n, followed by a translation along the x axis. The translation does not affect the scalar in Helmholtz's equation, so we will just call it C. Thus our transformation is

$$\begin{aligned}
X(x,y) &= nx + C \\
Y(x,y) &= ny.
\end{aligned}$$

An easier way to see the affect such a transformation would have on the scalar, consider the Jacobian matrix for the transformation

$$\mathcal{J} = \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}.$$

The determinant of this matrix is n^2, and gives us the desired result.

Note that if the original solution has rotational symmetry (i.e. is in the symmetry class A2), then the transformation will not change the symmetry class. This is easy to see since, when the equilateral triangle is subdivided into n^2 triangles, the resulting picture is rotationally symmetric. Since the solution inside each of these smaller triangles is the same rotationally symmetric symmetric solution, the resulting solution is rotationally symmetric.

However, if it started asymmetric (i.e. in the symmetry class E2) then this approach will lead to a solution in A2 iff n is divisible by 3. As noted above, the subdivided triangle is rotationally symmetric. However, since we are starting with a solution in E2, the resulting picture may or may not be rotationally symmetric. To test for rotational symmetry, consider Figure 7(b).

Looking at the solution inside the small triangle at the left of the subdivided triangle, we can sketch in the solution on the large subdivided triangle by reflecting this small triangle over the boundary and keep track of the nodal line. Comparing the small triangles along the top edge, we can see that after two reflections, the solutions appears rotated by ρ^{-1}.

Restricting ourselves to looking only at the 3 small triangles at the tips of a subdivided triangle, note that a rotation of the large triangle induces a rotation of the small triangles. This rotation only agrees with the reflection method if $\rho = \rho^{-(n-1)}$. Equivalently, $\rho^n = \mathbf{1}$, which means that n must be a multiple of 3.

Given a solution with E2 symmetry and scalar k^2, the harmonic with scalar 3^2k^2 is the same as the solution resulting from using the method from Section 4.1) twice $k^2 \rightarrow 3k^2 \rightarrow 9k^2$. It is easiest to see this by comparing the two pictures, shown in Figure 7. On the left, we have the picture using the harmonic method, and on the right we have the picture resulting from using the method from Section 4.1) twice. Following the two nodal patterns, we see that they are the same.

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